Nonparametric regression with irregular error distributions - an extreme value approach

(work in progress)

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Outline

1. Nonparametric regression models
   - Nonparametric mean and endpoint regression
   - (Ir)regularity of models
2. Regression estimators based on local maxima
   - Naive estimators
   - Local linear estimator
   - The general case
3. Application to analysis of error distribution

Nonparametric mean regression models

$$Y_i = g(x_i) + \varepsilon_i, \qquad 1 \le i \le n,$$

ε_i iid errors with $E(\varepsilon_i) = 0$

g “smooth” mean regression function

standard nonparametric regression estimators: local averages

$$\hat{g}(x) = \sum_i w(x - x_i) Y_i$$

with $w(t) \begin{cases} \text{large} \\ \text{small} \end{cases}$ if $|t| \begin{cases} \text{small} \\ \text{large} \end{cases}$

Estimators “optimal” if distribution of ε_i “regular”.

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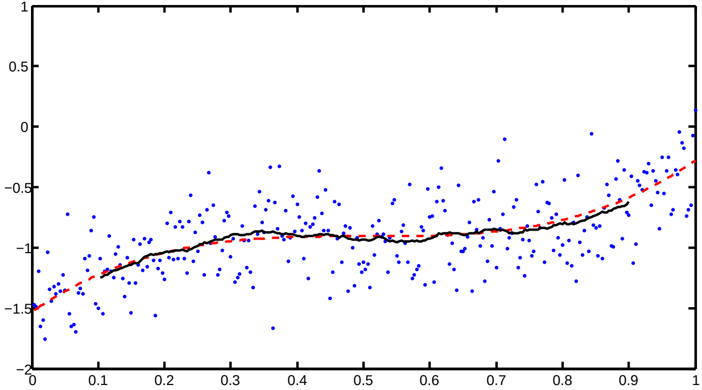
Estimators "optimal" if distribution of ε_i "regular".

Nonparametric mean regression models: Example

$\varepsilon_i = \tilde{\varepsilon}_i - E(\tilde{\varepsilon}_i)$ where $\tilde{\varepsilon}_i$ have cdf

$$F(y) = \mathrm{e}^{-|y|^{\alpha}}, \qquad y \le 0, \quad \alpha = 4$$

$g(x) = 5\left(x - \frac{1}{2}\right)^3$



Nonparametric boundary regression models

$$Y_i = g(x_i) + \varepsilon_i, \qquad 1 \le i \le n,$$

ε_i iid errors with right endpoint of support equal 0

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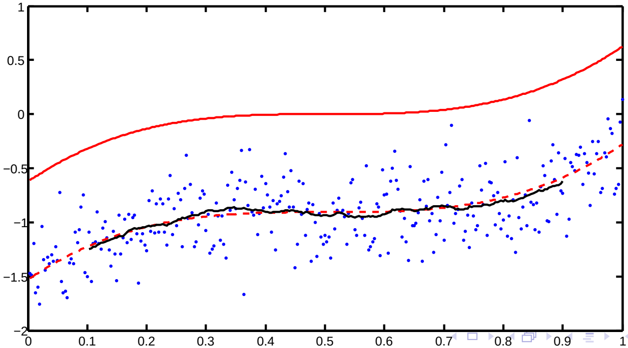

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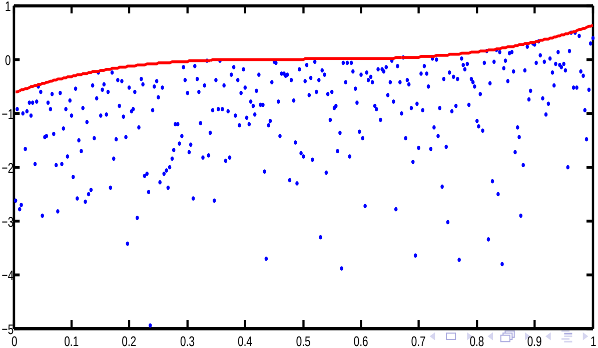
Example: ε_i with cdf $F(y) = \mathrm{e}^{-|y|^\alpha}, \qquad y \le 0, \quad \alpha = 1$



Model with equidistant design

$$Y_i = g\left(\frac{i}{n}\right) + \varepsilon_i, \qquad 1 \le i \le n,$$

ε_i iid with cdf $\quad F(y) = 1 - c|y|^\alpha + o\big(|y|^\alpha\big)$

α controls regularity:

$\alpha > 2$: regular model,
optimal estimators can be constructed from local averages (if error distribution known);
classical nonparametric rates of convergence

$\alpha < 2$: irregular model,
estimators based on local extrema yield better rates of convergence

$\alpha = 2$ critical value separating realm of regular models from irregular models

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(Ir)regularity from EVT point of view

$$Y_i = g\left(\frac{i}{n}\right) + \varepsilon_i, \qquad 1 \le i \le n,$$

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$\alpha > 2$	$\alpha < 2$
ML estimator of $\gamma = 1/\alpha$ asymptotically normal	ML estimator of γ not as. normal ($1 \le \alpha < 2$) or not even defined ($\alpha < 1$)
extreme quantile estimators based on EVT approximation outperform empirical quantiles	extreme quantile est. based on EVT approx. does not work well; empirical quantiles converge at faster rate

(Ir)regularity in more general experiments

General parametric model with observations

$$Y_i \text{ iid with density } f_\vartheta, \quad \vartheta \in \Theta \subset \mathbb{R}^d$$

w.r.t. some σ-finite measure μ.

Consider

$$s_\vartheta := \sqrt{f_\vartheta}$$

as element of Banach space $L_2(\mu)$.

Model is called regular if $\vartheta \mapsto s_\vartheta$ differentiable in $L_2(\mu)$.

Then

- ML estimator behaves nicely,
- characterization of optimal estimators and tests (in minimax sense or in sense of convolution theorem) well known.

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(Ir)regularity in location models

Special case:

$$Y_i = \vartheta + \varepsilon_i, \qquad 1 \leq i \leq n,$$

ε_i iid with density $f \quad \Longrightarrow \quad Y_i$ iid with density $f_\vartheta = f(\cdot - \vartheta)$

Model regular if f is sufficiently "smooth"

- regular models:
  - normal distribution
  - Weibull $f(y) = \alpha \exp(-|y|^{\alpha+1}) 1_{(-\infty,0)}(y)$ for $\alpha > 2$
- irregular models:
  - uniform distribution on $[\vartheta, 1+\vartheta]$ (or $[0,\vartheta]$ or $[-\vartheta,\vartheta]$)
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Nonparametric endpoint regression models

$$Y_i = g(x_i) + \varepsilon_i, \qquad 1 \leq i \leq n,$$

If g smooth, then regression model locally around fixed x similar to location model.

Heuristically, concept of (ir)regularity carries over.

Hence for

$$\varepsilon_i \text{ iid with cdf} \qquad F(y) = 1 - c|y|^\alpha + o\big(|y|^\alpha\big)$$

leads to

- regular model for $\alpha > 2$
- irregular model for $\alpha < 2$

Here mainly interested in uniform estimation of g in irregular model

Related literature

- Hall, Van Keilegom (2009): minimax rates (under quadratic loss) and minimax rate optimal estimators of $g(x)$ for fixed x
- Müller, Wefelmeyer (2010): similar results if error distribution is symmetric
- estimation of g based on iid observations (X_i, Y_i) with support $\{(x, y) \in [0, 1] \times [0, \infty) \mid y \leq g(x)\}$:

  Härdle, Park, Tsybakov (1995): special case of $\alpha = 1$, L_1-type loss
  Hall, Nussbaum, Stern (1997): conditional density with power behavior at boundary; asymptotics only for x fixed
- similar papers in Poisson process model
- Huge literature for models with monotone boundary function g (frontier estimation); problem much easier then.

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Naive estimator

Assume that regression fct. g belongs to following Hölder class

$$\mathcal{H}_{\beta,L} := \Big\{h : [0,1] \to \mathbb{R} \mid h \text{ is } s := \lfloor \beta - \rfloor \text{ times differentiable}, \; |h^{(s)}(x) - h^{(s)}(y)| \le L|x-y|^{\beta-s} \; \forall x, y \in [0,1]\Big\}$$

Approximate g locally in neighborhood of x by $g(x)$.

Then

$$Y_i \approx g(x) + \varepsilon_i$$

for all $i \in I_n(x) := \big\{i \in \{1, \ldots, n\} \mid |i/n - x| \le h_n\big\}$ for suitable $h_n \downarrow 0$.

Because $\varepsilon_i < 0$ but $\max_{i \in I_n(x)} \varepsilon_i \to 0$ in probability if $|I_n(x)| \sim 2nh_n \to \infty$, define

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Naive estimator: uniform asymptotics

$$\hat{g}_n(x) := \max_{i \in I_n(x)} Y_i \quad \text{with} \quad I_n(x) := \big\{ i \in \{1, \dots, n\} \mid |i/n - x| \le h_n \big\}$$

Theorem

If $h_n \to 0$, $nh_n \to \infty$, then

$$\sup_{x \in [h_n, 1-h_n]} |\hat{g}_n(x) - g(x)| = O(h_n^\beta) + O_P\left(\left(\frac{|\log h_n|}{nh_n}\right)^{1/\alpha}\right)$$

uniformly for all $g \in \mathcal{H}_{\beta,L}$ for all fixed $\beta \in (0,1]$, $L > 0$.

Bandwidth $h_n = (n/\log n)^{-1/(\alpha\beta+1)}$ leads to optimal rate $(n/\log n)^{-\beta/(\alpha\beta+1)}$;
faster than usual nonparametric rate $n^{-\beta/(2\beta+1)}(\log n)^\tau$ if $\alpha < 2$

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Proof

$$
\begin{aligned}
|\hat{g}_n(x) - g(x)| &= \Big| \max_{i \in I_n(x)} g(i/n) + \varepsilon_i - g(x) \Big| \\
&\leq \max_{i \in I_n(x)} |g(i/n) - g(x)| + \min_{i \in I_n(x)} |\varepsilon_i| \\
&= O(h_n^\beta) + \min_{i \in I_n(x)} |\varepsilon_i|
\end{aligned}
$$

uniformly in x. Let $k_n = \lfloor 2nh_n \rfloor - 1, \quad l_n = \lfloor n/k_n \rfloor \sim 1/(2h_n)$.

$$
\begin{aligned}
& P\Big\{ \max_{x \in [h_n, 1-h_n]} \min_{i \in I_n(x)} |\varepsilon_i| > u \Big\} \\
\leq \;& P\Big\{ \max_{1 \leq j \leq n} \min_{j \leq i \leq j+k_n} |\varepsilon_i| > u \Big\} \\
\leq \;& P\Big\{ \max_{0 \leq l \leq l_n} M_{n,l} > u \Big\} \\
\leq \;& P\Big\{ \max_{0 \leq l \leq l_n, l \text{ even}} M_{n,l} > u \Big\} + P\Big\{ \max_{0 \leq l \leq l_n, l \text{ odd}} M_{n,l} > u \Big\}
\end{aligned}
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with $\quad M_{n,l} := \max_{lk_n < j \leq (l+1)k_n} \min_{j \leq i \leq j+k_n} |\varepsilon_i|$.

Proof

$$\begin{aligned} |\hat{g}_n(x) - g(x)| &= \left| \max_{i \in I_n(x)} g(i/n) + \varepsilon_i - g(x) \right| \\ &\le \max_{i \in I_n(x)} |g(i/n) - g(x)| + \min_{i \in I_n(x)} |\varepsilon_i| \\ &= O(h_n^\beta) + \min_{i \in I_n(x)} |\varepsilon_i| \end{aligned}$$

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Proof (cont.)

Because $M_{n,l} = \max_{lk_n < j \le (l+1)k_n} \min_{j \le i \le j+k_n} |\varepsilon_i|$ are iid for l even

$$P\Big\{ \max_{0 \le l \le l_n, l \text{ even}} M_{n,l} > u \Big\} = 1 - \big(P\{M_{n,0} \le u\}\big)^{\lfloor l_n/2 \rfloor + 1},$$

and likewise for maximum over odd l.

If $M_{n,0} > u$, then consider first j such that $\min_{j \le i \le j+k_n} |\varepsilon_i| > u$:

$$\begin{aligned} P\{M_{n,0} > u\} &= P\Big\{ \min_{1 \le i \le 1+k_n} |\varepsilon_i| > u \Big\} + \sum_{j=2}^{k_n} P\Big\{ |\varepsilon_{j-1}| \le u, \min_{j \le i \le j+k_n} |\varepsilon_i| > u \Big\} \\ &= (1 - F_{|\varepsilon|}(u))^{k_n+1} + (k_n - 1) F_{|\varepsilon|}(u) \big(1 - F_{|\varepsilon|}(u)\big)^{k_n+1}. \end{aligned}$$

Combine everything to obtain

$$P\Big\{ \sup_{x \in [h_n, 1-h_n]} \min_{i \in I_n(x)} |\varepsilon_i| > u \Big\} \le 2\Big(1 - \Big(1 - (1 + k_n F_{|\varepsilon|}(u))(1 - F_{|\varepsilon|}(u))^{k_n}\Big)^{\lfloor l_n/2 \rfloor + 1}\Big)$$

from which the assertion easily follows.

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Naive estimator

Remark: Proof yields more precise asymptotics (not needed in what follows)

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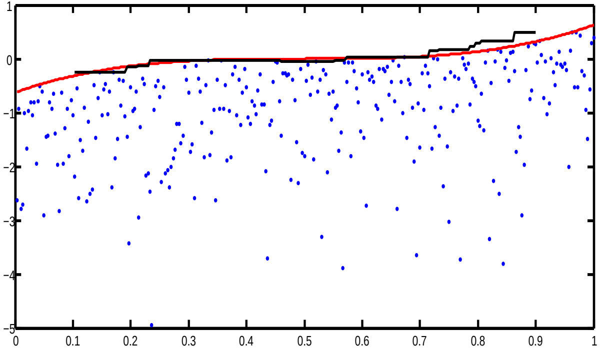

Performance not so good where function changes more quickly

Local linear approximation

If regression function g belongs to $\mathcal{H}_{\beta,L}$ for some $\beta > 1$ (i.e. it is smoother), then it can be locally approximated by linear function $g(t) \approx a_0 + a_1(t - x)$ for $|x - t| \leq h_n$;

approximation more accurate on same neighborhood of x, resp. larger neighborhood can be used to obtain same accuracy:

$$Y_i \approx a_0 + a_1\left(\frac{i}{n} - x\right) + \varepsilon_i \quad \text{if} \quad \left|\frac{i}{n} - x\right| \leq h_n,$$

where h_n can now be chosen larger.

Similar as before this leads to optimization problem to minimize a_0 under the constraints $a_0 + a_1\left(\frac{i}{n} - x\right) > Y_i$, i.e.

$$\hat{g}_n(x) := \min\left\{a_0 \in \mathbb{R} \mid \exists a_1 \in \mathbb{R} \text{ s.t. } a_0 + a_1\left(\frac{i}{n} - x\right) > Y_i \ \forall i \in I_n(x)\right\}.$$

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Further improvements for smoother functions

Suppose $g \in \mathcal{H}_{\beta,L}$ for some $\beta > 2$. Let $s = \lfloor \beta - \rfloor$. Then

$$g(t) \approx \sum_{j=0}^{s} a_j (t-x)^j$$

on neighborhood of x. However, for $x \notin \{1/n, 2/n, \dots, 1\}$

$$\tilde{g}_n(x) := \min\left\{ a_0 \in \mathbb{R} \mid \exists a_j \in \mathbb{R} \text{ s.t. } \sum_{j=0}^{s} a_j \left(\frac{i}{n} - x\right)^j > Y_i \ \forall i \in I_n(x) \right\} = -\infty$$

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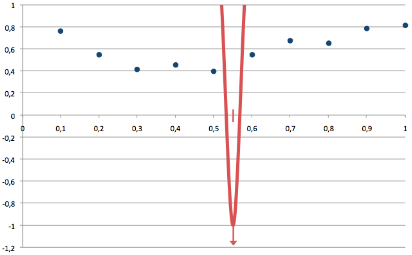

New interpretation of naive and local linear estimator

With

$$\mathcal{P}_s := \{p \text{ polynomial of order } s \text{ on } [x - h_n, x + h_n] \mid p(i/n) \geq Y_i \; \forall\, i \in I_n(x)\}$$

one has

$$\hat{g}_n(x) = p_s(x) \qquad \text{where} \qquad p_s \in \mathcal{P}_s \text{ minimizes } \int_{x-h_n}^{x+h_n} p(t)\, dt$$

for $s = 0$ (naive est.) resp. $s = 1$ (local linear est.)

Reason: $\int_{x-h_n}^{x+h_n} p(t)\, dt = 2p(x)h_n$ for $p \in \mathcal{P}_s$ with $s \in \{0, 1\}$

This interpretation can be easily generalized to higher order polynomials:

$$\hat{g}_{n,s}(x) = p_s(x) \qquad \text{with} \qquad p_s = \arg\min_{p_s \in \mathcal{P}_s} \int_{x-h_n}^{x+h_n} p(t)\, dt$$

Other variant:

$$\hat{g}^*_{n,s}(x) = p^*_s(x) \qquad \text{with} \qquad p^*_s = \arg\min_{p \in \mathcal{P}_s} \sum_{i \in I_n(x)} p(i/n)$$

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Uniform asymptotics of general estimators

$$\hat{g}_{n,s}(x) = p_s(x) \qquad \text{with} \qquad p_s = \arg\min_{p_s \in \mathcal{P}_s} \int_{x-h_n}^{x+h_n} p(t)\,dt$$

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Theorem

Suppose $g \in \mathcal{H}_{\beta,L}$ *for some* $\beta \in (s, s+1]$. *If* $h_n \to 0$, $nh_n \to \infty$, *then*

$$\sup_{x \in [h_n, 1-h_n]} |\hat{g}_{n,s}(x) - g(x)| = O(h_n^\beta) + O_P\left(\left(\frac{|\log h_n|}{nh_n}\right)^{1/\alpha}\right)$$

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uniformly for all $g \in \mathcal{H}_{\beta,L}$ *for all fixed* $L > 0$.

Bandwidth $h_n = (n/\log n)^{-1/(\alpha\beta+1)}$ leads to optimal rate $(n/\log n)^{-\beta/(\alpha\beta+1)}$

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Empirical process of residuals

Goal: Analyse distribution of errors (e.g. test for parametric hypotheses)

If errors ε_i were observable, statistical analysis based on empirical distribution function

$$F_n(y) := \frac{1}{n}\sum_{i=1}^{n} 1_{(-\infty,y]}(\varepsilon_i).$$

Use instead residuals

$$\hat{\varepsilon}_i := Y_i - \hat{g}_n(i/n), \qquad nh_n \le i \le n - nh_n,$$

with empirical distribution function

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Empirical process of residuals: asymptotics

$$\hat{F}_n(y) := \frac{1}{m_n} \sum_{i \in I_n} 1_{(-\infty, y]}(\hat{\varepsilon}_i), \qquad \tilde{F}_n(y) := \frac{1}{m_n} \sum_{i \in I_n} 1_{(-\infty, y]}(\varepsilon_i).$$

Theorem

If $\max_{i \in I_n} |\hat{g}_n(i/n) - g(i/n)| = O_P\big(n^{-\beta/(\alpha\beta+1)}(\log n)^\tau\big)$ *for some* $\tau < 0$ *and* F *is Hölder continuous of order* $\alpha \wedge 1$*, then*

$$\sup_{y \in \mathbb{R}} \big|\hat{F}_n(y) - \tilde{F}_n(y)\big| = o_P\big(n^{-1/2}\big).$$

if $1/\beta < \alpha < 2 - 1/\beta$*. In particular,*

$$\sqrt{n}(\hat{F}_n - F) \longrightarrow B \circ F$$

weakly in $D(\mathbb{R})$ *for a Brownian bridge* B*.*

Proof

Conditions on α ensure that $a_n := n^{-\beta/(\alpha\beta+1)+\varepsilon} = o\big(n^{-1/(2(\alpha\wedge 1))}\big)$ for sufficiently small $\varepsilon < 0$. By the assumption on $\hat{g}_n$

$$\begin{aligned}\hat{F}_n(y) &= \frac{1}{m_n}\sum_{i\in I_n} 1_{(-\infty,y]}(\hat{\varepsilon}_i) = \frac{1}{m_n}\sum_{i\in I_n} 1_{(-\infty,y]}\big(\underbrace{\varepsilon_i + g(i/n)}_{=Y_i} - \hat{g}_n(i/n)\big)\\ &\leq \frac{1}{m_n}\sum_{i\in I_n} 1_{(-\infty,y]}\big(\varepsilon_i - a_n\big) = \tilde{F}(y+a_n)\end{aligned}$$

with probability tending to 1. Now

$$\sup_{y\in\mathbb{R}}(F(y+a_n) - F(y)) = O\big(a_n^{\alpha\wedge 1}\big) = o(n^{-1/2}).$$

and

$$E_n := \sqrt{n}(\tilde{F}_n - F) \to B \circ F$$

where the limit has continuous sample paths. Hence, uniformly for all $y \in \mathbb{R}$,

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Proof (cont.)

Likewise,

$$\sqrt{n}(\hat{F}_n(y) - \tilde{F}(y)) \geq E_n(y - a_n) - E_n(y) + \sqrt{n}(F(y - a_n) - F(y)) = o_p(1),$$

which concludes the proof.

Further results and open problems

Working group at HU Berlin: established estimators which adapt to unknown smoothness and attain pointwise minimax rates.

Open problems to be considered next in working group at Hamburg University:

- More precise analysis of estimation error to obtain similar results for empirical process of residuals under weaker conditions on α
- Analysis of tail empirical process of residuals
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Thank you for your attention!