(Extremes and related properties of) Stochastic asymmetric Lagrange waves

Georg Lindgren, Finn Lindgren, Sofia Åberg, . . .

Lund University, University of Bath, Lund University

Lisbon, 18 – 20 March 2013
Recent advances in Extreme value Theory
honoring Ross Leadbetter

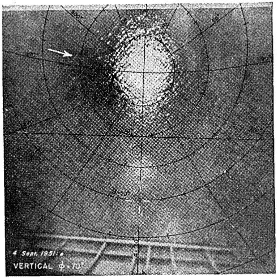
Contents

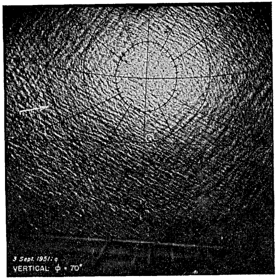
A wall of water

An experiment 62 years ago

Measurement of the Roughness of the Sea Surface from Photographs of the Sun's Glitter

CHARLES COX AND WALTER MUNK
Scripps Institution of Oceanography, La Jolla, California*
(Received April 28, 1954)



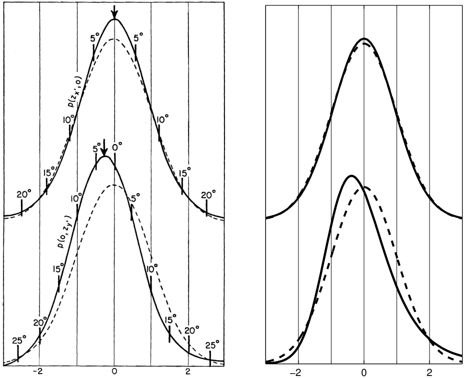


The Cox and Munk experiment 1951–1954 on wave asymmetry –

compared to Lagrange asymmetry; left – Cox/Munk, right – Lagrange

top – across waves

bottom – along waves



Some good literature

Proc. Camb. Phil. Soc. (1966), **62**, 263 263
PCPS 62–39
Printed in Great Britain

Local maxima of stationary processes

By M. R. LEADBETTER
Research Triangle Institute, Durham, N. Carolina, U.S.A.

(*Received* 15 *July* 1965)

Abstract. Two natural definitions for the distribution function of the height of an 'arbitrary local maximum' of a stationary process are given and shown to be equivalent. It is further shown that the distribution function so defined has the correct frequency interpretation, for an ergodic process. Explicit results are obtained in the normal case.

"The correct frequency interpretation"

What is the height of "an arbitrary local maximum" in a stationary (differentiable) stochastic process $X(t)$ – and what is its distribution?

Answer: The limit of the empirical distribution function as observation interval grows.

$$
\begin{aligned}
&P(\text{height of a local maximum} \leq u) \\
&= \lim_{T\to\infty} \frac{T^{-1}\#\text{local maxima in } [0, T] \text{ with height less than } u}{T^{-1}\#\text{local maxima in } [0, T]} \\
&= \frac{E(\#\{t \in [0,1], X'(t) = 0, \text{downcrossing, AND } X(t) \leq u\})}{E(\#\{t \in [0,1], X'(t) = 0, \text{downcrossing}\})}
\end{aligned}
$$

Also David Slepian 1962.

Rice formula gives answers

Rice formula gives the expectation of the number of crossings and the number of "marked" crossings per time unit of a stationary process; explicit for Gaussian processes.

- The slope at the instance of upcrossings of any level has a Rayleigh distribution
- The height of a local maximum is distributed as

$$\alpha U + \sqrt{1-\alpha^2}R; \quad U \sim N(0,1), R \sim \text{Rayleigh}$$

where α measures the width of the energy spectrum

The Gaussian wave model (1952)

The height $W(t,s)$ of the water surface at location s at time t is a Gaussian stationary (homogeneous) random process, expressed as a sum (integral !)

$$W(t,s) = \sum_k A_k \cos(\kappa_k s - \omega_k t + \phi_k)$$

of moving cosines with

- random amplitudes A_k
- random phases ϕ_k

and with fixed frequencies (1/wave period) and wave numbers (1/wave length).

Gaussian characteristics

A Gaussian sea is statistically symmetric:

- the sea surface can be turned upside down
- a wave movie can be run backwards

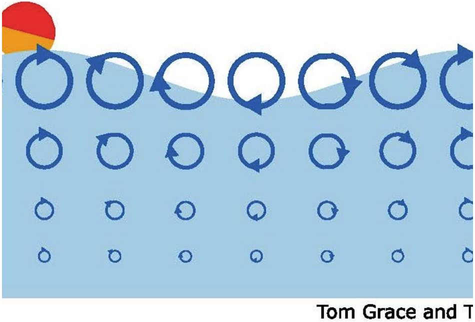
and you don't see any difference

It needs to be combined with physics/hydrodynamics – gives the Lagrange model

Why using a stochastic Lagrange model?

- The (linear) Gaussian wave model allows exact computation of wave characteristic distributions (the WAFO toolbox)
- produces crest-trough and front-back stochastically symmetric waves
- The modified stochastic Lagrange model can produce
  (2006) crest-trough asymmetric waves (2D)
  (2009) front-back asymmetric waves (2D)
  (2011) front-back asymmetric waves (3D)
- still allows for exact computation of wave characteristic distributions
- Software exists for calculation of statistical wave characteristic distributions (crest, trough heights, period, steepness etc)

Waves according to Google search



Circles in deep water, ellipses in shallow water

TASK: Combine Gauss with Google

Gaussian generator and the orbital spectrum

In the Gaussian model the vertical height $W(t, \boldsymbol{s})$ of a particle at the free surface at time t and location $\boldsymbol{s} = (u, v)$ is an integral of harmonics with random phases and amplitudes:

$$\boldsymbol{\kappa} = (\kappa_x, \kappa_y) = \kappa(\cos\theta, \sin\theta)$$
$$\omega = \omega(\boldsymbol{\kappa}) = \sqrt{g\kappa \tanh \kappa h}$$
$$W(t, \boldsymbol{s}) = \int_{\omega=0}^{\infty} \int_{\theta=-\pi}^{\pi} e^{i(\boldsymbol{\kappa s} - \omega t)}\, d\zeta(\omega, \theta)$$

with $S(\omega, \theta) =$ the "orbital spectrum" and $\zeta(\omega, \theta)$ is a Gaussian complex "spectral process".

Wave direction $= \theta$

The stochastic Lagrange model –

Describes horizontal and vertical movements of individual surface water particles. Use

$$W(t, \mathbf{s}) = \int e^{i(\boldsymbol{\kappa}\mathbf{s} - \omega t)} \, d\zeta(\boldsymbol{\kappa}, \omega)$$

for the vertical movement of a particle with (intitial) reference coordinate $\mathbf{s} = (u, v)$ and write

$$\mathbf{\Sigma}(t, \mathbf{s}) = \begin{pmatrix} X(t, \mathbf{s}) \\ Y(t, \mathbf{s}) \end{pmatrix} = \text{horizontal location at time } t$$

$\mathbf{\Sigma}(t, \mathbf{s}) - \mathbf{s}$ is the horizontal discplacement of a particle from its original location $\mathbf{s}$

– with horizontal Gaussian movements

Use the same (vertical) Gaussian spectral process as in $W(t, \boldsymbol{s})$ to generate also the horizontal variation

Fouques,Krogstad,Myrhaug,Socquet-Juglard (2004), Gjøsund (2000,2003) Aberg, Lindgren, Lindgren (2006,2007,2008,2009,2011), Guerrin (2009)

$$\boldsymbol{\Sigma}(t, \boldsymbol{s}) = \begin{pmatrix} X(t, \boldsymbol{s}) \\ Y(t, \boldsymbol{s}) \end{pmatrix} = \boldsymbol{s} + \int \boldsymbol{H}(\theta, \kappa)\, e^{i(\boldsymbol{\kappa} \boldsymbol{s} - \omega t)}\, d\zeta(\boldsymbol{\kappa}, \omega)$$

where the filter function $\boldsymbol{H}$ depends on water depth h:

$$\boldsymbol{H}(\theta, \kappa) = i\, \frac{\cosh \kappa h}{\sinh \kappa h} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$$

The stochastic Lagrange model

The 3D stochastic first order Lagrange wave model is the triple of Gaussian processes

$$(W(t,s), \mathbf{\Sigma}(t,s)) = (W(t,s), X(t,s), Y(t,s))$$

All covariance functions and auto-spectral and cross-spectral density functions for $\mathbf{\Sigma}(t,s)$ follow from the orbital spectrum $S(\omega,\theta)$ and the filter equation.

Space wave : keep time coordinate fixed

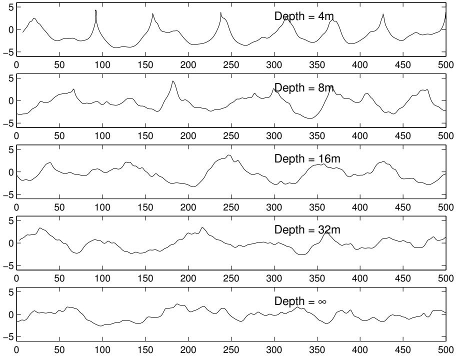
Time wave : keep space coordinate – $(X(t,s), Y(t,s))$ – fixed

This will (later) lead us to a "Palm type" problem – what are the distributions of $(W(t,s), \mathbf{\Sigma}^{-1}(t,(0,0)))$ when $(X(t,s), Y(t,s)) = (0,0)$, say.

Lagrange 2D space waves, time fixed

A Lagrange space wave at time t_0 is the parametric curve (2D) or surface (3D)

$$L(x) : \mathbf{s} \Rightarrow (X_M(t_0, \mathbf{s}), W(t_0, \mathbf{s}))$$



Ocean waves need a direction

The free Lagrange waves are peaked –

but they have no preferred direction!

Front-back asymmetric Lagrange waves

To get realistic front-back asymmetry one needs a model with external input from wind, for example, by a parameter α:

$$\frac{\partial^2 X(t,s)}{\partial t^2} = \frac{\partial^2 X_M(t,s)}{\partial t^2} + \alpha W(t,s)$$

The filter function from vertical $W(t,u)$ to horizontal $X(t,u)$ is then

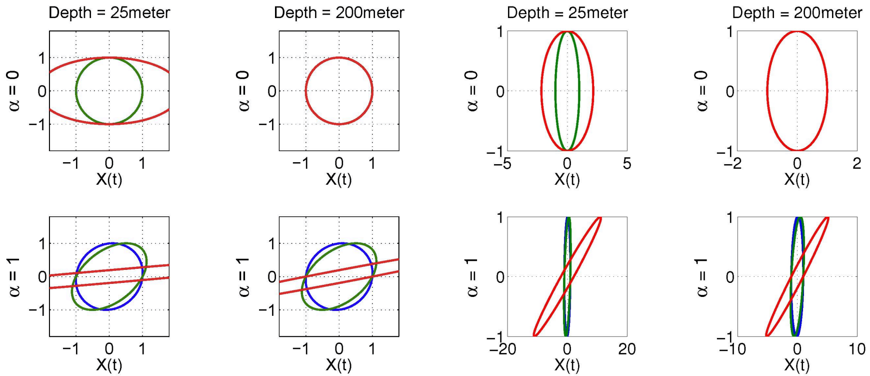
$$H(\omega) = i\,\frac{\cosh \kappa h}{\sinh \kappa h} + \frac{\alpha}{(-i\omega)^2} = \rho(\omega)\, e^{i\theta(\omega)},$$

Implies an extra phase shift ($\theta = \pi/2$ in the free model)

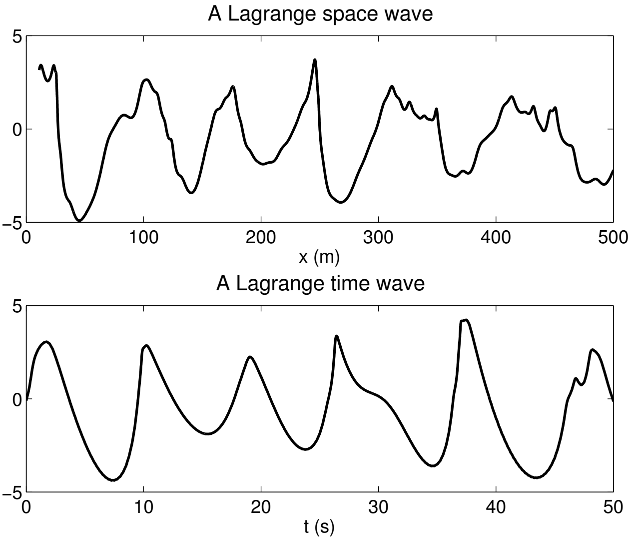
$$X(t,u) = u + \int e^{i(\kappa u - \omega t + \theta(\omega))} \rho(\omega)\, d\zeta(\omega,\kappa)$$

Particle path depends on wavelength

Wavelength: 1m (blue), 5m (green), 50m (red)



Space and time waves



For 3D waves the filter function needs to take directional spreading into account

An example:

$$\boldsymbol{H}(\theta, \kappa) = \frac{\alpha}{(i\omega)^2} \cdot \begin{pmatrix} \cos^2(\theta)\,|\cos(\theta)| \\ \cos^2(\theta)\sin(\theta)\,\mathrm{sign}(\cos\theta) \end{pmatrix} + i\frac{\cosh \kappa h}{\sinh \kappa h} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

to take care of wind blowing in positive x-direction.

Front-back asymmetry depends on directional spreading in the orbital spectrum

Explicit definitions – I

The model

$$(W(t,s), \mathbf{\Sigma}(t,s)) = (W(t,s), X(t,s), Y(t,s))$$

is an implicitly defined model. The space and time models can be made explicit by

$$\boldsymbol{s} = \mathbf{\Sigma}^{-1}(t,(x,y))$$

equal to the reference point that is mapped to the observation point (x,y).

Note: There may be many solutions to $\mathbf{\Sigma}(t,\boldsymbol{s}) = (x,y)$.

Explicit definitions – II

Space wave $L(t_0, (x, y)) = W(t_0, \mathbf{\Sigma}^{-1}(t_0, (x, y)))$
$=$ photo of the surface

Time wave $L(t, (x_0, y_0))$ is the parametric curve:

$$t \mapsto W(t, \mathbf{\Sigma}^{-1}(t, (x_0, y_0)))$$

$=$ measured at a wave pole

Wave slopes in space (Cox/Munk) and time

The Cox/Munk experiment observed wave slopes at fixed time – needs space derivatives:

$$\begin{pmatrix} W_u \\ W_v \end{pmatrix} = \begin{pmatrix} X_u & Y_u \\ X_v & Y_v \end{pmatrix} \begin{pmatrix} L_x \\ L_y \end{pmatrix} \tag{1}$$

An oil platform experiences wave slopes at fixed location – needs time derivatives

$$\begin{aligned} W_t &= L_t + \begin{pmatrix} X_t & Y_t \end{pmatrix} \begin{pmatrix} L_x \\ L_y \end{pmatrix} \\ L_t &= W_t - \begin{pmatrix} W_u & W_v \end{pmatrix} \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \end{pmatrix}^{-1} \begin{pmatrix} X_t \\ Y_t \end{pmatrix} \end{aligned} \tag{2}$$

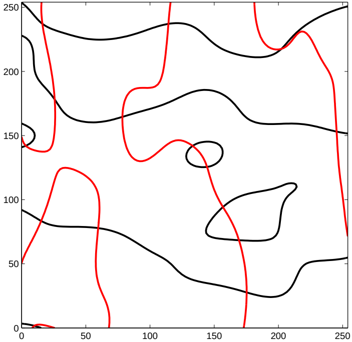
Asynchronous sampling in space = Cox/Munch, $t_0 = 0$

To find the distributions of the slopes L_x, L_y at a fixed point, say $(0,0)$, invert (1) and find

the distribution of $(W_u, W_v, X_u, X_v, Y_u, Y_v)$ at reference point $\mathbf{s}$

under the condition that $(X(\mathbf{s}), Y(\mathbf{s})) = (0,0)$

Contour lines prefer to cross under straight angles; "perpendicularitybias"



Standard Rice formulas

The original Rice formula gives the expected number of level crossings of level u per time unit in a stationary process:

$$
\begin{aligned}
E(\#\{t \in [0,1]; X(t) &= u, \text{upcrossing}\}) \\
&= E(X'(0)^+ \mid X(0) = u)\, f_{X(0)}(u)
\end{aligned}
$$

Marked Rice formula

The Rice formula for marked crossing gives the expectede number of level crossings per time unit in a stationary process, at which a "mark" has a certain characteristic. The "mark" can be e.g. the derivative at the upcrossing.

$$
\begin{aligned}
&E(\#\{t \in [0,1]; X(t) = u, \text{upcrossing, and "mark"} \in B\}) \\
&\quad = E(X'(0)^{+} \times 1_{\text{"mark"} \in B} \mid X(0) = u)\, f_{X(0)}(u)
\end{aligned}
$$

Multivariate Rice formula

A generalized (multivariate) Rice formula gives the expected number of simultaneous crossings with marks, for example with marks

$$\text{mark:}(W_u, W_v, X_u, X_v, Y_u, Y_v)]_{\boldsymbol{s}}$$

with

$$\text{simultanous crossings: } (X(\boldsymbol{s}), Y(\boldsymbol{s})) = (0, 0)$$

Reference: Azaïs and Wschebor: Level sets and extrema of random processes and fields, Wiley, 2009.

Application for slope distribution with asynchronous sampling

Consider u-level crossings at $(0, 0)$ in a specified direction. Slope is defined as a function of $(W_u, W_v, X_u, X_v, Y_u, Y_v)]_{\mathbf{s}}$. With

$$N_0(A) = \#\{\mathbf{s} \in R^2; \mathbf{\Sigma}(\mathbf{s}) = (0,0), \text{slope} \in A\}$$

$$E(N_0(A)) = \underbrace{\int_{R^2} E\left(\underbrace{|\det \mathbf{\Sigma}'(\mathbf{s})|} \times 1_{\text{slope}\in A} \mid \mathbf{\Sigma}(\mathbf{s}) = (0,0)\right)}_{\text{perpendicularitybiascorrection}} f_{\mathbf{\Sigma}(\mathbf{s})}(0,0)\, d\mathbf{s}$$

The expectation is simply obtained by simulation.

Some old literature – still very useful

Proc. Camb. Phil. Soc. (1966), **62**, 263 263
PCPS 62–39
Printed in Great Britain

Local maxima of stationary processes

By M. R. LEADBETTER
Research Triangle Institute, Durham, N. Carolina, U.S.A.

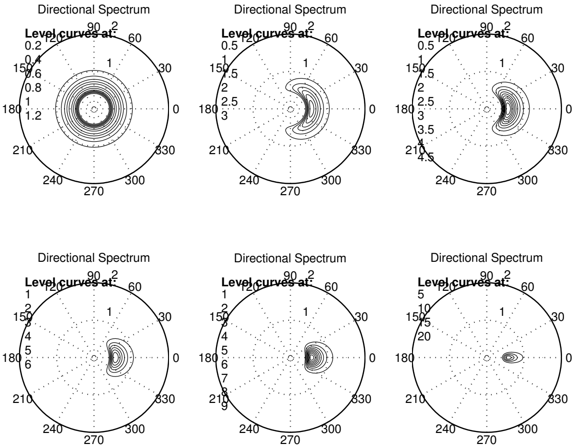
(*Received* 15 *July* 1965)

Abstract. Two natural definitions for the distribution function of the height of an 'arbitrary local maximum' of a stationary process are given and shown to be equivalent. It is further shown that the distribution function so defined has the correct frequency interpretation, for an ergodic process. Explicit results are obtained in the normal case.

Lesson to be learned

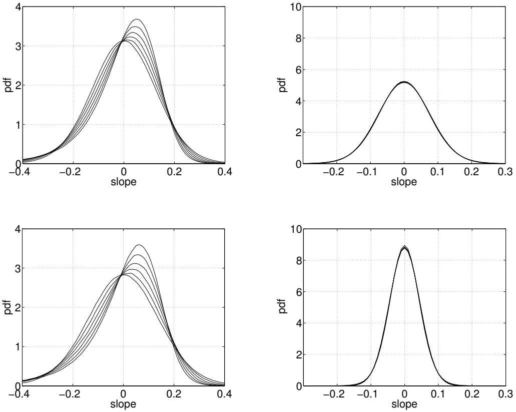
The ratio between the expected values of marked and unmarked crossings is equal to the empirical observable distribution.

Example specta - Pierson-Moskowitz with different spreading



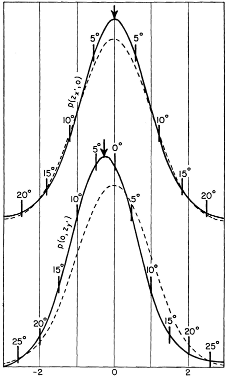
Slope PDF across wind and along wind, asynchrone

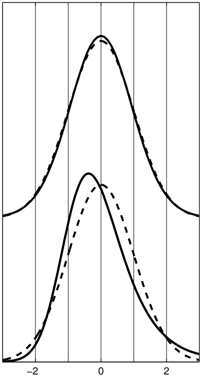
Top: Large spreading – Bottom: Little spreading
Left: Along wind – Right: Across wind; cf. Cox and Munk



The Cox and Munk experiment 1951–1954 on wave asymmetry –

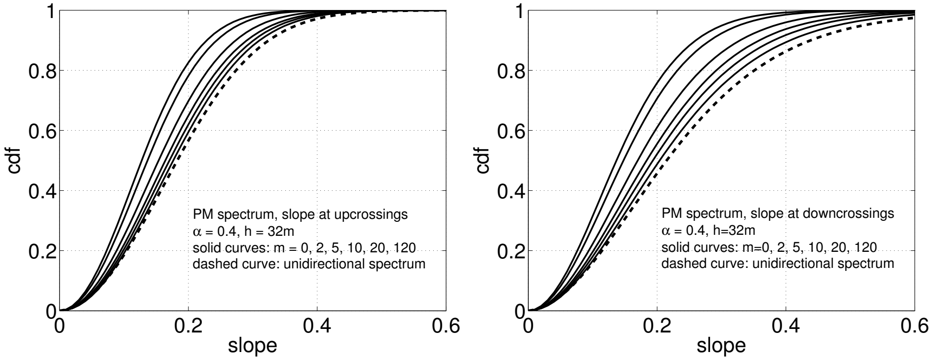
compared to Lagrange asymmetry:





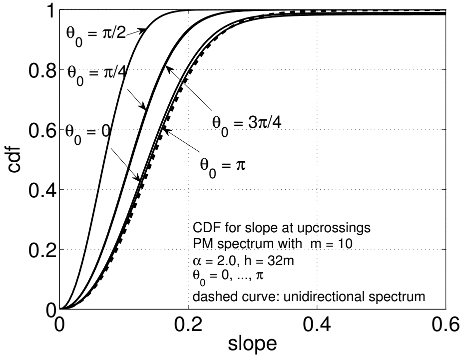
Slope CDF for different directional spreading

Left: slope at upcrossings – Right: Slope CDF at downcrossings
Moderate linkage parameter: $\alpha = 0.4$



Slope CDF for different heading; Strong linkage: $\alpha = 2$

Left: slope at upcrossings – Right: Slope CDF at downcrossings



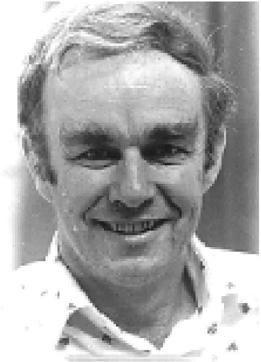
Summary

- The Gaussian wave model for the free water surface gives stochastically symmetric waves – trough/crest and front/back
- A first order Lagrange model for the move of individual waterparticles models vertical and horizontal movements as correlated Gaussian processes
- The transfer function between vertical and horizontal movements is based on hydrodynamic theory with a 90^o phase shift
- The "free" Lagrange model gives crest/trough asymmetric waves
- A modified Lagrange model with frequency dependent phase shift gives also front/back asymmetry
- Distributions of slopes and other wave characteristcs can be found by the generalized Rice formula

References

- Aberg, S. (2007). Wave intensities and slopes in Lagrangian seas. *Adv. Appl. Prob.*, **39**
- Lindgren, G. (2006). Slepian models for the stochastic shape of individual Lagrange sea waves. *Adv. Appl. Prob.* **38**
- Lindgren, G. (2009). Exact asymmetric slope distributions in stochastic Gauss-Lagrange ocean waves. *Appl. Ocean Res*, **31**
- Lindgren, G. (2010). Slope distribution in front-back asymmetric stochastic Lagrange time waves. *Adv. Appl. Prob.*, **42**
- Lindgren, G. and Aberg, S. (2008). First order stochastic Lagrange models for front-back asymmetric ocean waves. *J. OMAE*, **131**
- Lindgren, G. and Lindgren, F. (2011). Stochastic asymmetry properties of 3D-Gauss-Lagrange ocean waves with directional spreading. *Stochastic Models*, **27**
- WAFO-group, (2011). *WAFO – a Matlab Toolbox for Analysis of Random Waves and Loads; Tutorial for WAFO Version 2.5*, URL http://www.maths.lth.se/matstat/wafo

From all of us to Ross



Thank you for

- good theory
- good advice
- good company

We couldn't have been lead better