Multivariate tail representations

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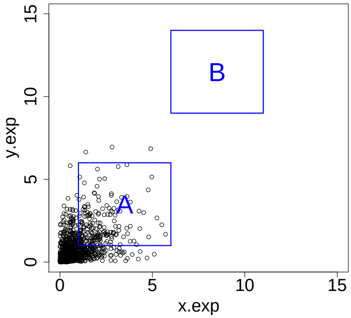
Motivation

- Goal: estimate $\mathrm{P}(\mathbf{X} \in B)$
- Method: Link $\mathrm{P}(\mathbf{X} \in B)$ to $\mathrm{P}(\mathbf{X} \in A)$ exploiting decay structure of extreme events



- Theoretical characterizations allow extrapolation of multivariate tail
- No natural direction of extrapolation in multivariate space

Approaches to multivariate extremes

- Interested in the extremes of a random vector $\mathbf{X} = (X_1, \ldots, X_d)$
- What is the multivariate tail of $\mathbf{X}$?

Approaches:

1. All components extreme
   - Asymptotic dependence
   - Asymptotic independence
2. At least one component extreme

This talk

- Review some existing representations
- Explore an alternative one

Notation

Focus on $d = 2$. Define

$$
\begin{array}{lll}
(X_P, Y_P) & \text{Pareto}(1) & \qquad\qquad \mathrm{P}(X_P > x) = x^{-1} \\
(X_E, Y_E) & \text{Exponential}(1) & \qquad\qquad \mathrm{P}(X_E > x) = e^{-x}
\end{array}
$$

Joint distribution / survivor functions:

$$
\begin{aligned}
F_P(x, y) &:= \mathrm{P}(X_P \leq x, Y_P \leq y) \\
\bar{F}_P(x, y) &:= \mathrm{P}(X_P > x, Y_P > y)
\end{aligned}
$$

All components extreme

- Study the behaviour of (X_P, Y_P) as they grow at the same rate
- i.e. what can we say about

$$F_P(nx, ny)$$

or

$$\bar{F}_P(nx, ny)$$

as $n \to \infty$?

- Key is assumption of multivariate regular variation

de Haan and Resnick (1977); Ledford and Tawn (1997)

All components extreme

Limiting extremal dependence described by

$$\lim_{n\to\infty} n[1 - F_P(nx, ny)] = \lim_{n\to\infty} n[(nx)^{-1} + (ny)^{-1} - \bar{F}_P(nx, ny)]$$
$$= V(x, y)$$

$V(x, y)$ homogeneous order -1.

Asymptotic dependence if

$$\lim_{n\to\infty} n\bar{F}_P(nx, ny) > 0$$

Asymptotic independence if

$$\lim_{n\to\infty} n\bar{F}_P(nx, ny) = 0, \text{i.e.}$$
$$\lim_{n\to\infty} n[1 - F_P(nx, ny)] = x^{-1} + y^{-1}.$$

All components extreme: asymptotic independence

Limit tells us nothing:

$$\lim_{n\to\infty} n\bar{F}_P(nx, ny) = 0.$$

Sub-asymptotic theory gives rate of convergence to zero limit:

$$\bar{F}_P(nx, ny) = n^{-1/\eta} L(nx, ny)(xy)^{-1/2\eta}$$

- $\eta \in (0, 1]$ coefficient of tail dependence
- $L(x, y)$ bivariate slowly varying: $\lim_{n\to\infty} L(nx, ny)/L(n, n) = g(x, y)$, homogeneous order 0.

Ledford and Tawn (1996, Bka; 1997, JRSSB); Resnick (2002, Extremes); Ramos and Ledford (2009, JRSSB)

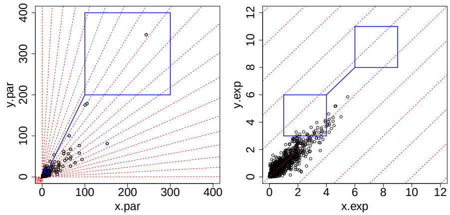
Extrapolation strategies: all components extreme

Assumption gives asymptotic link between probabilities:

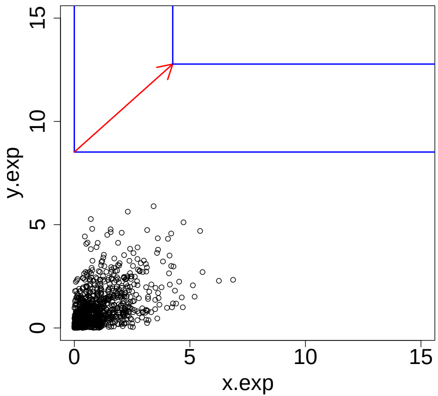
$$\begin{aligned} \mathrm{P}\{(X_P, Y_P) \in tnA\} &\sim t^{-1/\eta}\mathrm{P}\{(X_P, Y_P) \in nA\} \\ \mathrm{P}\{(X_E, Y_E) \in t + n + A\} &\sim \exp\{-t/\eta\}\mathrm{P}\{(X_E, Y_E) \in n + A\} \end{aligned}$$

(Asymptotic dependence if $\eta = 1$). i.e. extrapolate on

- rays emanating from the origin in Pareto margins
- lines parallel to the diagonal in exponential margins



Weakness of this approach



Need for alternatives

- May not have large values of X occurring with large values of Y
- Theories where we break away from the idea of both components growing at the same rate can lead to different extrapolation strategies

One component extreme: conditional extremes

- Let one component (Y, say) be extreme: how should X grow in relation to get an interesting limit?

Assume there exist normalisation functions a, b s.t.

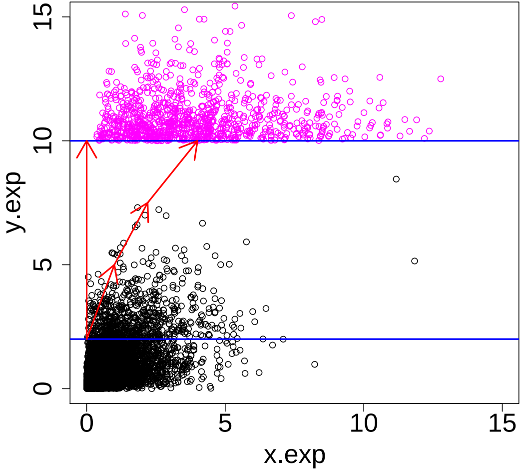
$$\mathrm{P}\left(\frac{X_E - b(Y_E)}{a(Y_E)} \leq x, Y_E - z > y \,\middle|\, Y_E > z\right) \to H(x)\exp\{-y\},$$

as $z \to \infty$, where H is a non-degenerate distribution function.

- Pro: Assumption holds widely; extrapolate according to form of a, b
- Con: Do not know a, b or H: statistical estimation necessary, inference messy

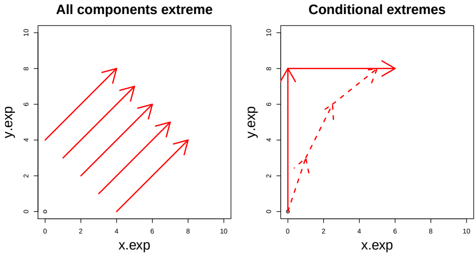
Heffernan and Tawn (2004, JRSSB); Heffernan and Resnick (2007, Ann. App. Prob.)

Extrapolation strategies: one component extreme



Alternative asymptotic theory

Different trajectories from different theories. In exponential margins:



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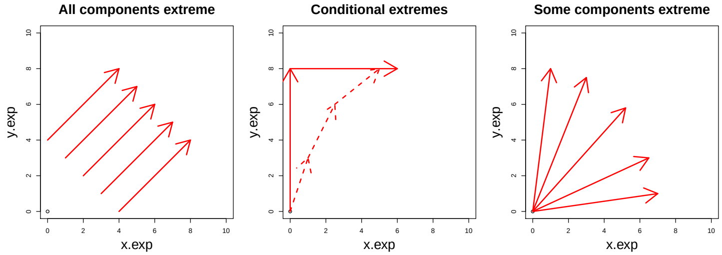


Alternative: some components more extreme than others

- fix different growth rates of X, Y
- examine induced joint tail probability decay rate. Focus on

$$\bar{F}_P(n^\beta, n^\gamma) \equiv \bar{F}_E(\beta \log n, \gamma \log n), \quad \beta, \gamma \geq 0, \beta \wedge \gamma > 0$$

Different marginal growth rates

Basic assumption

For all $\beta, \gamma \geq 0$, $\beta \wedge \gamma > 0$,

$$\bar{F}_P(n^\beta, n^\gamma) = n^{-\kappa(\beta,\gamma)} L(n; \beta, \gamma)$$

as $n \to \infty$.

- $\kappa(\beta, \gamma)$ homogeneous of order 1. Maps different marginal growth rates to joint tail decay rate
- $L(n; \beta, \gamma)$ univariate slowly varying in n for all $\beta, \gamma \geq 0, \beta \wedge \gamma > 0$: $\lim_{n\to\infty} L(nt; \beta, \gamma)/L(n; \beta, \gamma) = 1$

Different marginal growth rates: exploiting homogeneity

Basic assumption

For all $\beta, \gamma \geq 0$, $\beta \wedge \gamma > 0$,

$$\bar{F}_P(n^{\beta}, n^{\gamma}) = n^{-\kappa(\beta,\gamma)} L(n; \beta, \gamma)$$

as $n \to \infty$.

- Homogeneity of κ: only relative marginal growth rates relevant
- Let $\omega \in [0,1]$. Define $\lambda(\omega) := \kappa(\omega, 1-\omega)$, termed angular dependence function

$$\bar{F}_P(n^{\omega}, n^{1-\omega}) = n^{-\lambda(\omega)} L(n; \omega, 1-\omega).$$

Regular variation

- Key idea (as ever) is regular variation
- Assume a univariate regular variation condition to study multivariate tails
- Let $T_\omega : [1,\infty) \mapsto [1,\infty)^2$ be given by $T_\omega(x) = (x^\omega, x^{1-\omega})$. Then we study $\bar{F}_P$ such that

$$\bar{F}_P \circ T_\omega(n) \in RV_{-\lambda(\omega)}$$

- Replace single multivariate regular variation condition by whole class of univariate ones

Some properties of λ

- Marginal condition: $\lambda(0) = \lambda(1) = 1$
- Range under non-negative association of (X, Y):

$$\max\{\omega, 1-\omega\} \leq \lambda(\omega) \leq 1$$

- Independence: $\lambda(\omega) = 1$
- Asymptotic dependence: $\lambda(\omega) = \max\{\omega, 1-\omega\}$
- Coefficient of tail dependence: $\eta^{-1} = 2\lambda(1/2)$

More on λ

- λ plays a role like Pickands' dependence function,

$$A(w) := V\left(\frac{1}{1-w}, \frac{1}{w}\right),$$

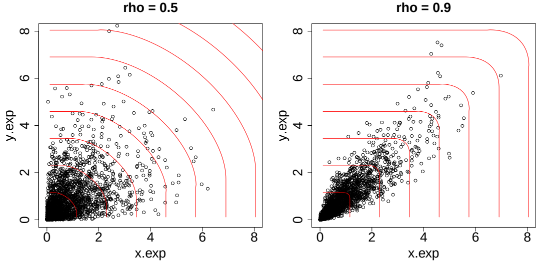
  but for asymptotically independent distributions

- $\lambda(\omega) \equiv \max\{\omega, 1-\omega\}$ when $A(w)$ takes 'interesting' forms (asymptotic dependence)
- $A(w) \equiv 1$ when λ takes 'interesting' forms (asymptotic independence)
- Convexity of λ entails an additional dependence condition

Examples

Bivariate normal

$$\lambda(\omega) = \begin{cases} \frac{1-2\rho\omega^{1/2}(1-\omega)^{1/2}}{(1-\rho^2)}, & \rho^2 < \min\left\{\frac{\omega}{1-\omega}, \frac{1-\omega}{\omega}\right\} \\ \max\{\omega, 1-\omega\}, & \text{otherwise} \end{cases}$$



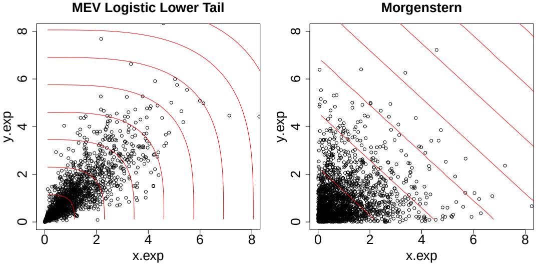
Examples

Lower joint tail of MV extreme value distribution

$\lambda(\omega) = V(1/\omega, 1/1-\omega) = A(1-\omega)$ [A: Pickands' dependence function]

Morgenstern

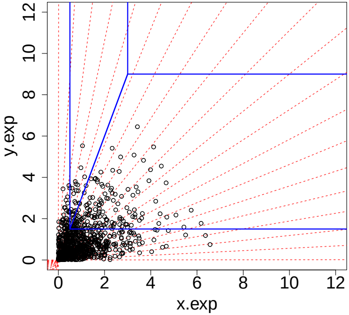
$\lambda(\omega) = 1$



Extrapolation strategies: some components extreme

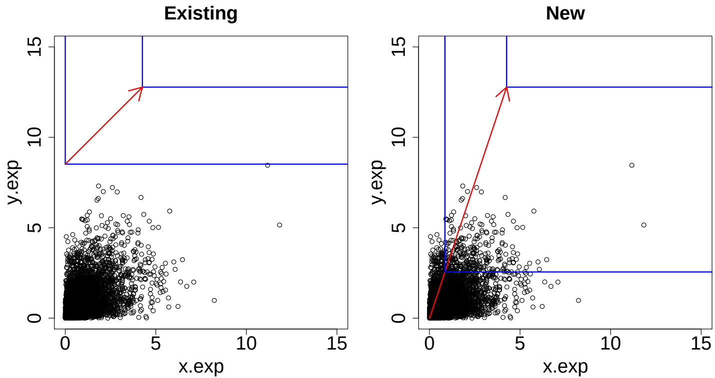
- Focus on joint survivor set
- As $v \to \infty$,

$$\mathsf{P}\{X_E > \omega(t+v), Y_E > (1-\omega)(t+v)\} \sim \exp\{-\lambda(\omega)t\}\mathsf{P}\{X_E > \omega v, Y_E > (1-\omega)v\}$$



Extrapolate upon rays emanating from the origin in exponential margins

Advantage



Estimating $\lambda(\omega)$

- Estimation of $\lambda(\omega)$ key to estimation of any probability on ray ω
- Regularly varying tail: use Hill estimator

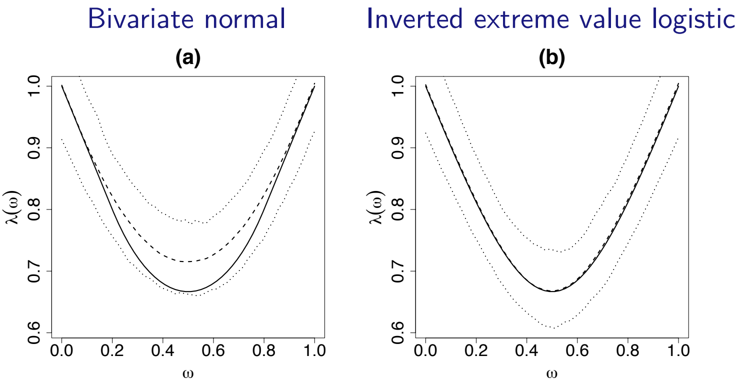
Let $Z_\omega = \min\left\{\frac{X_E}{\omega}, \frac{Y_E}{1-\omega}\right\}$, then

$$\mathrm{P}(Z_\omega > \log n + \log x) = L(nx; \omega, 1-\omega)(nx)^{-\lambda(\omega)}, \quad n \to \infty$$

- For $k+1$ top order statistics of Z_ω, $z_{(1)}, \ldots, z_{(k+1)}$:

$$\hat{\lambda}(\omega) = \left(\frac{1}{k}\sum_{i=1}^{k}(z_{(i)} - z_{(k+1)})\right)^{-1}$$

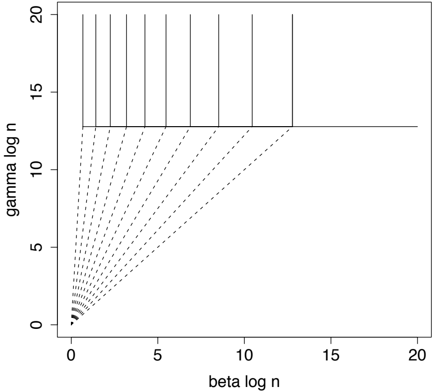
Example estimated $\lambda(\omega)$



- $\eta = 0.75$ in each case
- Solid line: true; dashed / dotted lines: pointwise mean and 95% CI from 500 repetitions
- SV function affects finite sample estimation for bivariate normal

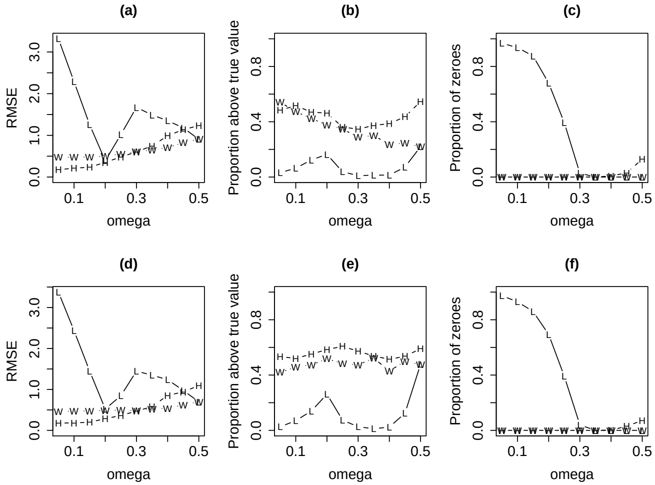
Estimation quality

- How well can we estimate probabilities in different parts of the quadrant?
- Evaluate methods of: Ledford and Tawn, Heffernan and Tawn, and current method for $\omega \in \{0.05, 0.1, \ldots, 0.5\}$



Estimation quality

Ledford and Tawn (L), Heffernan and Tawn (H), and current method (W)



Multivariate theory

Can we develop an asymptotic link between more general sets than joint survivor?

Consider

$$\bar{F}_P(n^{\omega}x, n^{1-\omega}y)$$

as $n \to \infty$.

Some theoretical progress possible with assumptions:

1. κ differentiable at $(\omega, 1-\omega)$
2. $\lim_{n\to\infty} \frac{L\left(n;\omega+\frac{\log x}{\log n},1-\omega+\frac{\log y}{\log n}\right)}{L(n;\omega,1-\omega)} = 1$

Multivariate theory

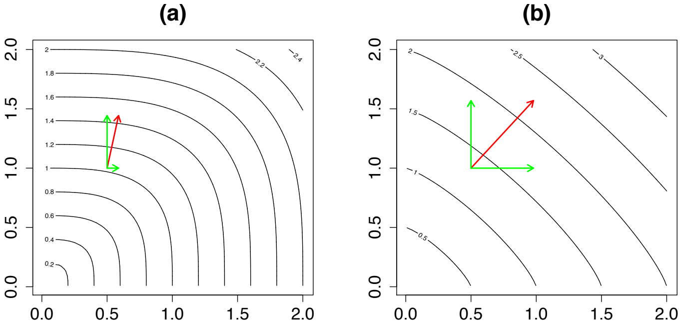
Convergence of conditional probability under these assumptions:

$$\mathrm{P}(X_P > n^{\omega}x, Y_P > n^{1-\omega}y \mid X_P > n^{\omega}, Y_P > n^{1-\omega}) \to x^{-\kappa_1(\omega)}y^{-\kappa_2(\omega)},$$

$x, y \geq 1$.

- $\kappa_1, \kappa_2 \geq 0$ are partial derivatives of κ, evaluated at $(\omega, 1-\omega)$
- $\kappa_1 = \lambda(\omega) + (1-\omega)\lambda'(\omega)$; $\kappa_2 = \lambda(\omega) - \omega\lambda'(\omega)$
- stochastically independent limit, with parameters determined by form of finite-level dependence

Multivariate theory



Multivariate theory

General form of convergence

$$\mathrm{P}\left\{\left(\frac{X_P}{n^{\omega}}, \frac{Y_P}{n^{1-\omega}}\right) \in \cdot \,\middle|\, \left(\frac{X_P}{n^{\omega}}, \frac{Y_P}{n^{1-\omega}}\right) \in [1,\infty)^2\right\} \xrightarrow{w} \pi(\cdot;\omega)$$

- Limit measure $\pi(\cdot;\omega)$ homogeneous of order $-\{\kappa_1(\omega)+\kappa_2(\omega)\}$

Multivariate theory

General form of convergence

$$\mathrm{P}\left\{\left(\frac{X_P}{n^{\omega}}, \frac{Y_P}{n^{1-\omega}}\right) \in \cdot \,\middle|\, \left(\frac{X_P}{n^{\omega}}, \frac{Y_P}{n^{1-\omega}}\right) \in [1,\infty)^2\right\} \xrightarrow{w} \pi(\cdot;\omega)$$

- Limit measure $\pi(\cdot;\omega)$ homogeneous of order $-\{\kappa_1(\omega)+\kappa_2(\omega)\}$
- Provides asymptotic link between Borel sets $B \subset [1,\infty)^2$

$$\mathrm{P}\left\{\left(\frac{X_P}{n^{\omega}}, \frac{Y_P}{n^{1-\omega}}\right) \in (t^{\omega}, t^{1-\omega})B\right\} \sim t^{-\lambda(\omega)}\mathrm{P}\left\{\left(\frac{X_P}{n^{\omega}}, \frac{Y_P}{n^{1-\omega}}\right) \in B\right\},$$

$t > 1$.

- Follows since

$$\omega\kappa_1(\omega) + (1-\omega)\kappa_2(\omega) = \kappa(\omega, 1-\omega) = \lambda(\omega)$$

by homogeneity

Multivariate theory: assumptions

- Assumptions of differentiable κ and smoothness of L do not hold under asymptotic dependence when $\omega = 1/2$
- Here,

$$\mathrm{P}(X_P > n^{1/2}x, Y_P > n^{1/2}y \mid X_P > n^{1/2}, Y_P > n^{1/2})$$

$\rightarrow$

$$\frac{x^{-1} + y^{-1} - V(x,y)}{2 - V(1,1)}$$

as $n \to \infty$.

- All results can be combined under a generalized notion of multivariate regular variation of the random vector

Back to multivariate regular variation

- Define the bijective map $U_\omega : (1,\infty)^2 \mapsto (1,\infty)^2$ by

$$U_\omega(x,y) = (x^\omega, y^{1-\omega}),$$

for any fixed $\omega \in (0,1)$

- Assume $\bar{F}_P \circ U_\omega$ is multivariate regularly varying of index $-\lambda(\omega)$

Back to multivariate regular variation

- Define the bijective map $U_\omega : (1,\infty)^2 \mapsto (1,\infty)^2$ by

$$U_\omega(x,y) = (x^\omega, y^{1-\omega}),$$

  for any fixed $\omega \in (0,1)$

- Assume $\bar{F}_P \circ U_\omega$ is multivariate regularly varying of index $-\lambda(\omega)$
- Equivalently: standard multivariate regular variation of the law of the random vector $U_\omega^{\leftarrow}(X_P, Y_P) = (X_P^{1/\omega}, Y_P^{1/(1-\omega)})$, $\omega \in (0,1)$

$$\mathrm{P}\left\{U_\omega^{\leftarrow}(X_P, Y_P)/n \in \cdot \mid U_\omega^{\leftarrow}(X_P, Y_P)/n \in [1,\infty)^2\right\} \xrightarrow{w} \pi^*(\cdot;\omega)$$

- $\pi^*(cB;\omega) = c^{-\lambda(\omega)}\pi^*(B;\omega)$, for Borel $B \subset (1,\infty)^2$, $c > 0$.

Generalizing the scaling

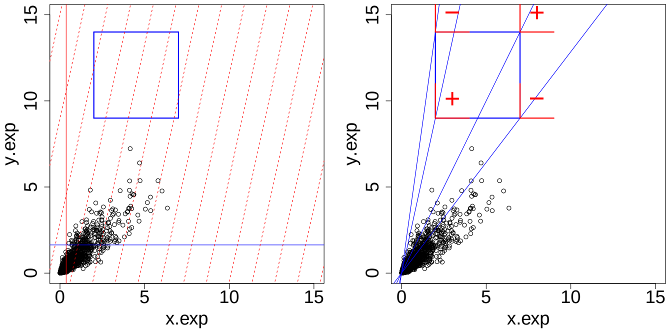
- In place of conditional probability can generalize the scaling
- We can find $a(n) \in RV_{1/\lambda(\omega)}$ such that for Borel $B \subset (1, \infty)^2$,

$$\lim_{n\to\infty} n\mathrm{P}\{U_\omega^{\leftarrow}(X_P, Y_P)/a(n) \in B\} = \pi^*(B;\omega)$$

  with $\pi^*(\partial B) = 0$ and π^* as previous slide.
- If $\bar{H}_\omega$ is the survivor function of $\min\{U_\omega^{\leftarrow}(X_P, Y_P)\}$ then take $a(n) = (1/\bar{H}_\omega)^{\leftarrow}(n)$
- For $\omega = 1/2$ we recognize this as standard multivariate regular variation; $a(n) \in RV_{1/2\eta}$
- Extension for $\omega \neq 1/2$, under additional assumptions

Multivariate theory: consequences

- In theory multivariate case opens up links between $\mathrm{P}(\mathbf{X} \in B)$ and $\mathrm{P}(\mathbf{X} \in A)$ for general A, B
- How best to exploit? Which ω? Depends on shape of set, extrapolation direction, ...?



Links to existing theory

Natural link to the 'all components extreme' approach:

- Assume (a version of) the existing regular variation theory holds for transformed variables

$$(X_P^{1/\omega}, Y_P^{1/(1-\omega)}) \quad \text{or equivalently} \quad (X_E/\omega, Y_E/(1-\omega))$$

- Univariate regular variation condition:

$$\text{Ledford \& Tawn (1996)} : \bar{F}_P(n,n) \in RV_{-1/\eta}$$
$$\text{Here} : \bar{F}_P \circ T_\omega(n,n) \in RV_{-\lambda(\omega)}$$

- Multivariate regular variation condition:

$$\text{Ledford \& Tawn (1997)} : \bar{F}_P(nx,ny) \text{ MVRV index } -1/\eta$$
$$\text{Here} : \bar{F}_P \circ U_\omega(nx,ny) \text{ MVRV index } -\lambda(\omega)$$

Links to existing theory

- Ramos and Ledford (2009) explore limits of the type

$$\mathrm{P}(X_P > nx, Y_P > ny \mid X_P > n, Y_P > n) \to g(x,y)(xy)^{-1/2\eta}$$

- There, the focus is on characterization of $g(x,y)$, or equivalently the hidden angular measure H_η
- Analogous to characterization of $\pi^*(B; 1/2)$ in

$$\lim_{n\to\infty} n\mathrm{P}\{U_\omega^{\leftarrow}(X_P, Y_P)/a(n) \in B\} = \pi^*(B;\omega)$$

for both asymptotically dependent and asymptotically independent possibilities

Links to existing theory

Link to conditional limit theory? A much more modest connection...

$$\mathrm{P}(X_P > \beta \log n | Y_P > \log n) = L(n; \beta, 1) \exp\{-[\kappa(\beta, 1) - 1] \log n\}$$

- Some smoothness conditions required as β becomes $\beta(n, x)$
- Limit depends on the behaviour of the SV function $L(n; \omega, 1 - \omega)$ as $\omega \to 0$ or $\omega \to 1$
- Full link requires further characterization / assumptions on L

Higher dimensions

- Representation extends to d-dimensions: for $\boldsymbol{\omega} \in S_{d-1}$: with $S_{d-1} := \{\mathbf{v} \in [0,1]^d : \sum_{i=1}^{d} v_i = 1\}$

$$\bar{F}_P(n^{\omega_1}, \ldots, n^{\omega_d}) = L(n; \boldsymbol{\omega}) n^{-\lambda(\boldsymbol{\omega})}, \quad n \to \infty.$$

- Complication as always is the number of different orders that can arise with

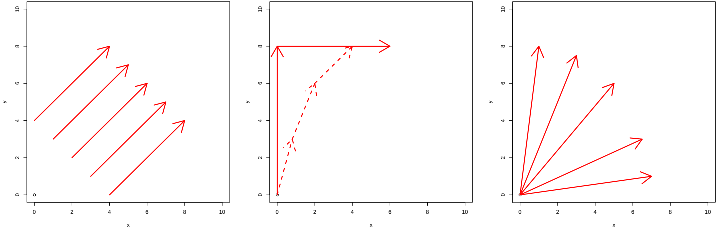
$$\bar{F}_P(nx_1, nx_2), \quad \bar{F}_P(nx_1, nx_3), \quad \bar{F}_P(nx_2, nx_3), \quad \bar{F}_P(nx_1, nx_2, nx_3), \ldots$$

- Ideal: a representation we can estimate in all d dimensions, which provides accurate inference for all lower dimensions

- Challenging without simplifying assumptions on asymptotic (in)dependence of different orders

To summarize

- Alternative asymptotic theories lead to different extrapolation directions
- Theory based upon margins growing at different rates offers alternative direction to previous theories



- Simple univariate regular variation condition gives theory for extrapolating joint survivor sets
- Some multivariate theory available with additional assumptions

Thanks for your attention!