SYMPOSIUM ON RECENT ADVANCES IN EXTREME VALUE THEORY HONORING ROSS LEADBETTER,

Properties of max-stable random fields: conditional distrubutions and strong mixing.

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Structure of the talk

1. Introduction
2. Conditional distribution of max-stable random fields
3. Strong mixing properties of max-i.d. processes

Motivations

- Needs for modeling spatial extremes in environmental sciences :
  - maximal temperatures in a heat wave,
  - intensity of winds during a storm,
  - water heights in a flood ...
- Spatial extreme value theory - geostatistics of extremes :
  - ▷ Schlather ('02), Models for stationary max-stable random fields.
  - ▷ de Haan & Pereira ('06), Spatial extremes : Models for the stationary case.
  - ▷ Davison, Ribatet & Padoan ('11), Statistical modelling of spatial extremes.
- Max-stable random fields play a crucial role as possible limits of normalized pointwise maxima of i.i.d. random fields.

Notion of max-stable random field

Let T be a compact metric space (usually, $T \subset \mathbb{R}^2$).
Let $\eta = (\eta(t))_{t\in T}$ be a continuous process on T.

Definition

- η is called max-stable if for all $n \geq 1$, there are continuous functions $a_n(\cdot) > 0$ and $b_n(\cdot)$ such that

$$\left(\frac{\bigvee_{i=1}^n \eta_i(t) - b_n(t)}{a_n(t)}\right)_{t\in T} \stackrel{\mathcal{L}}{=} \big(\eta(t)\big)_{t\in T}$$

  where $\eta_1, \ldots, \eta_n$ are independent copies of η.
- η is called max-i.d. if for all $n \geq 1$, there exists independent continuous processes $(\eta_{i,n})_{1\leq i\leq n}$ such that

$$\Big(\bigvee_{i=1}^n \eta_{i,n}(t)\Big)_{t\in T} \stackrel{\mathcal{L}}{=} \big(\eta\big)_{t\in T}.$$

Notion of max-stable random field

- References : Resnick '87, de Haan '84, de Haan & Pickands '86, Giné Hahn & Vatan '90, Resnick & Roy '91, Penrose '92, Schlather '02, de Haan & Ferreira '06, $\cdots$
- Max-stable processes arrise as weak limit of (normalized) maxima of i.i.d. processes :

$$\text{If} \quad a_n^{-1}(\vee_{i=1}^n X_i - b_n) \Longrightarrow \eta, \quad \text{then } \eta \text{ max-stable.}$$

  Max-i.d. processes arrise as weak limit of maxima from a triangular array of independent processes :

$$\text{If} \quad a_n^{-1}(\vee_{i=1}^n X_{i,n} - b_n) \Longrightarrow \eta, \quad \text{then } \eta \text{ max-i.d.}$$

- Max-stable processes play a central role in functional extreme value theory as Gaussian process in functional statistics.

Structure of the talk

1 Introduction

2 Conditional distribution of max-stable random fields

3 Strong mixing properties of max-i.d. processes

Motivations

- Observations of a max-stable process η at some stations only :

$$\eta(s_i) = y_i, \quad i = 1, \ldots, k. \tag{O}$$

  How to predict what happens at other locations ?
- We are naturally lead to consider the conditional distribution of η given the observations (O).
- Different goals :
  - theoretical formulas for the conditional distribution,
  - sample from the conditional distribution,
  - compute (numerically) the conditional median or quantiles ...
- Results for spectrally discrete max-stable processes :

  ▷ Wang & Stoev ('11), Conditional sampling of spectrally discrete max-stable processes.
- New results for max-stable and even max-i.d. processes.

Structure of max-i.d. processes

Theorem (de Haan '84, Giné Hahn & Vatan '90)

For any continuous, max-i.d. random process $\eta = (\eta(t))_{t\in T}$ satisfying $\mathrm{ess\,inf}\,\eta(t) \equiv 0$, there exists a unique Borel measure μ on $\mathcal{C}_0 = \mathcal{C}(T,[0,+\infty)) \setminus \{0\}$ such that

$$\Big(\eta(t)\Big)_{t\in T} \overset{\mathcal{L}}{=} \Big(\bigvee_{\phi\in\Phi} \phi(t)\Big)_{t\in T}, \quad \text{with } \Phi \sim \mathrm{PPP}(\mu).$$

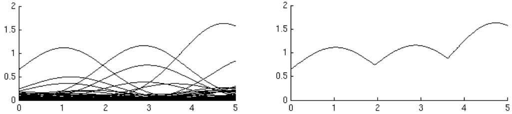
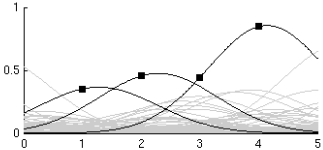


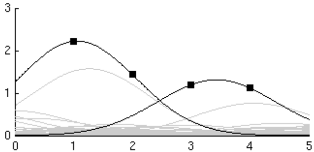
FIGURE: A realization of Φ and $\eta = \max(\Phi)$ for Smith 1D storm process.

Hitting scenario and extremal functions

- Observations $\{\eta(s_i) = y_i,\ 1 \leq i \leq k\}$ with $\eta(s) = \bigvee_{\phi \in \Phi} \phi(s)$.
- Assume that the law of $\eta(s_i)$ has no atom, $1 \leq i \leq k$.
  Then, with probability 1, $\exists!\ \phi_i \in \Phi, \quad \phi_i(s_i) = \eta(s_i)$.
- Definition of the following random objects :
  - the hitting scenario Θ, a partition of $S = \{s_1, \ldots, s_k\}$ with ℓ blocks,
  - the extremal functions $\varphi_1^+, \cdots, \varphi_\ell^+ \in \Phi$,
  - the subextremal functions $\Phi_S^- \subset \Phi$.
- Example with $k = 4$:



$\Theta = (\{s_1\}, \{s_2\}, \{s_3, s_4\})$



$\Theta = (\{s_1, s_2\}, \{s_3, s_4\})$

Joint distribution

Let $\mathcal{P}_k$ be the set of partitions of $\{s_1,\cdots,s_k\}$. We note $\mathbf{s}=(s_1,\ldots,s_k)$.

Theorem

For $\tau=(\tau_1,\ldots,\tau_\ell)\in\mathcal{P}_k$, $A\subset\mathcal{C}_0^\ell$ and $B\subset M_p(\mathcal{C}_0)$ measurable

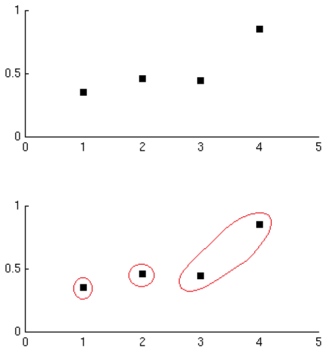
$$\begin{aligned}
&\mathbb{P}\Big[\Theta=\tau,\ (\varphi_1^+,\ldots,\varphi_\ell^+)\in A,\ \Phi_S^-\in B\Big]\\
=&\int_{\mathbb{C}_0^\ell} 1_{\{\forall j\in[\![1,\ell]\!],\ f_j>_{\tau_j}\vee_{j'\neq j}f_{j'}\}}1_{\{(f_1,\cdots,f_\ell)\in A\}}\\
&\qquad\mathbb{P}[\{\Phi\in B\}\cap\{\forall\phi\in\Phi,\ \phi<_S\vee_{j=1}^\ell f_j\}]\,\mu(df_1)\cdots\mu(df_\ell)
\end{aligned}$$

Furthermore, the law of $\eta(\mathbf{s})$ is equal to $\nu_{\mathbf{s}}=\sum_{\tau\in\mathcal{P}_k}\nu_{\mathbf{s}}^\tau$ with

$$\begin{aligned}
\nu_{\mathbf{s}}^\tau(d\mathbf{y}) &:= \mathbb{P}[\eta(\mathbf{s})\in d\mathbf{y},\ \Theta=\tau]\\
&= \mathbb{P}[\eta(\mathbf{s})\leq\mathbf{y}]\bigotimes_{j=1}^{\ell}\mu\Big(f(\mathbf{s}_{\tau_j^c})<\mathbf{y}_{\tau_j^c},\ f(\mathbf{s}_{\tau_j})\in d\mathbf{y}_{\tau_j}\Big).
\end{aligned}$$

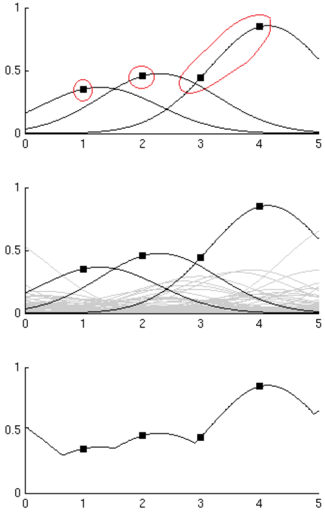
Conditional distribution

A three step procedure for the conditional law of η given $\eta(\mathbf{s}) = \mathbf{y}$:



Step 1 : sample Θ from the conditional law w.r.t. $\eta(\mathbf{s}) = \mathbf{y}$.

Conditional distribution



Step 2 : sample (φ_j^+) from the conditional law w.r.t. $\eta(\mathbf{s}) = \mathbf{y}$, $\Theta = \tau$.

Step 3 : sample $\Phi_{\mathbf{s}}^-$ from the conditional law w.r.t. $\eta(\mathbf{s}) = \mathbf{y}$, $\Theta = \tau$, $(\varphi_j^+) = (f_j)$.

Finally, set
$\eta(t) = \bigvee_{j=1}^{|\Theta|} \varphi_j^+(t) \bigvee_{\phi \in \Phi_{\mathbf{s}}^-} \phi(t)$.

Main theorem

Theorem

Let $\mathbf{s} \in T^k$ and $\mathbf{y} \in (0, +\infty)^k$.

1. For all $\tau \in \mathcal{P}_K$,

$$\mathbb{P}[\Theta = \tau \mid \eta(\mathbf{s}) = \mathbf{y}] = \frac{d\nu_{\mathbf{s}}^{\tau}}{d\nu_{\mathbf{s}}}(\mathbf{y}).$$

2. Conditionally on $\eta(\mathbf{s}) = \mathbf{y}$ and $\Theta = \tau$, the extremal functions $(\varphi_j^+)_{1 \le j \le |\tau|}$ are independent and φ_j^+ has distribution

$$\mu\big(df \mid f(\mathbf{s}_{\tau_j}) = \mathbf{y}_{\tau_j}, f(\mathbf{s}_{\tau_j^c}) < \mathbf{y}_{\tau_j^c}\big).$$

3. The conditional distribution of $\Phi_{\mathbf{s}}^-$ given $\eta(\mathbf{s}) = \mathbf{y}$, $\Theta = \tau$ and $(\varphi_j^+) = (f_j)$ is a PPP with intensity $1_{\{f(\mathbf{s}) < \mathbf{y}\}}\mu(df)$.

The regular case

- The model is called *regular* at $\mathbf{s}$ if

$$\mu(f(\mathbf{s}) \in d\mathbf{y}) = \lambda_{\mathbf{s}}(\mathbf{y})\, d\mathbf{y}.$$

- For a regular model,

$$\begin{aligned} \nu_{\mathbf{s}}^{\tau}(d\mathbf{y}) &:= \mathbb{P}[\eta(\mathbf{s}) \in d\mathbf{y},\ \Theta = \tau] \\ &= \mathbb{P}[\eta(\mathbf{s}) \leq \mathbf{y}] \Big(\prod_{j=1}^{|\tau|} \int_{\{\mathbf{u}_j < \mathbf{y}_{\tau_j^c}\}} \lambda_{(\mathbf{s}_{\tau_j}, \mathbf{s}_{\tau_j^c})}(\mathbf{y}_{\tau_j}, \mathbf{u}_j) d\mathbf{u}_j \Big)\, d\mathbf{y}, \end{aligned}$$

$$\mathbb{P}[\Theta = \tau \mid \eta(\mathbf{s}) = \mathbf{y}] = \frac{1}{C(\mathbf{s}, \mathbf{y})} \prod_{j=1}^{|\tau|} \int_{\{\mathbf{u}_j < \mathbf{y}_{\tau_j^c}\}} \lambda_{(\mathbf{s}_{\tau_j}, \mathbf{s}_{\tau_j^c})}(\mathbf{y}_{\tau_j}, \mathbf{u}_j) d\mathbf{u}_j,$$

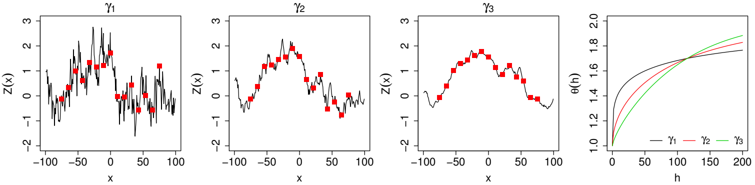
$$\mathbb{P}[\varphi_j^+(\mathbf{t}) \in d\mathbf{z} \mid \eta(\mathbf{s}) = \mathbf{y}, \Theta = \tau] = \frac{1}{C'(\mathbf{s}, \mathbf{y}, \tau)} \Big(\int_{\mathbf{u}_j < \mathbf{y}_{\tau_j^c}} \lambda_{(\mathbf{s}_{\tau_j}, \mathbf{s}_{\tau_j^c}, \mathbf{t})}(\mathbf{y}_{\tau_j}, \mathbf{u}_j, \mathbf{z}) d\mathbf{u}_j \Big) d\mathbf{z}.$$

Application to Brown-Resnick processes

- A Brown-Resnick process is a 1-Fréchet max-stable process

$$\eta(t) = \bigvee_{i \geq 1} U_i e^{W_i(t) - \gamma(t)/2}$$

  with $\{U_i, i \geq 1\} \sim \mathrm{PPP}(u^{-2}du)$ and $(W_i)_{i\geq 1}$ i.i.d. copies of a Gaussian process with stationary increment and variogram γ.
- Brown-Resnick processes driven by FBM :



Application to Brown-Resnick processes

- If $\mathrm{Var}[W(\mathbf{t})]$ is nonsingular, η is regular at $\mathbf{t}$:

$$\lambda_{\mathbf{t}}(\mathbf{z}) = C_{\mathbf{t}} \exp\left(-\frac{1}{2}\log \mathbf{z}^T Q_{\mathbf{t}} \log \mathbf{z} + L_{\mathbf{t}}^T \log \mathbf{z}\right) \prod_{i=1}^{k} z_i^{-1}$$

  with $C_{\mathbf{t}}$, $Q_{\mathbf{t}}$, $L_{\mathbf{t}}$ functions of $\mathrm{Var}[W(\mathbf{t})]$.
- Conditional sampling :
  - Step 1 : explicit formula for conditional hitting scenario distribution, but for large k combinatorial explosion of the state space $\mathcal{P}_k$.
  - Step 2 : extremal functions are conditioned log-normal processes.

Gibbs sampler

- The conditional hitting scenario distribution

$$\mathbb{P}[\Theta = \tau \mid \eta(\mathbf{s}) = \mathbf{y}] = \frac{1}{C(\mathbf{s},\mathbf{y})} \prod_{j=1}^{|\tau|} \int_{\{\mathbf{u}_j < \mathbf{y}_{\tau_j^c}\}} \lambda_{(\mathbf{s}_{\tau_j}, \mathbf{s}_{\tau_j^c})}(\mathbf{y}_{\tau_j}, \mathbf{u}_j) d\mathbf{u}_j,$$

  is of the form

$$\pi(\tau) = \frac{1}{C} \prod_{j=1}^{|\tau|} \omega_{\tau_j}, \quad \tau = (\tau_1, \ldots, \tau_\ell)$$

  with $C = \sum_{\tau \in \mathcal{P}_k} \prod_{j=1}^{|\tau|} \omega_{\tau_j}$.

- Combinatorial explosion for large k :

$$\sharp\mathcal{P}_5 = 52, \quad \sharp\mathcal{P}_{10} \approx 1,16 \cdot 10^5, \quad \sharp\mathcal{P}_{20} \approx 5,17 \cdot 10^{14}.$$

  We cannot evaluate C ! !

Gibbs sampler

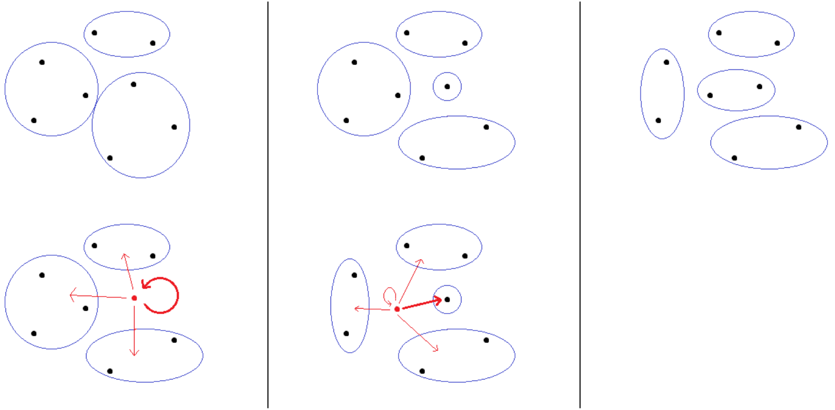
- Gibbs sampler : irreducible and aperiodic Markov Chain (Θ_n) on $\mathcal{P}_k$ with invariant law π.
- Transition kernel based on the conditional distribution of the partition « with $k-1$ points frozen » :

$$\mathbb{P}[\Theta_{n+1} = \tau' \mid \Theta_n = \tau] = \sum_{i=1}^{k} \frac{1}{k} \pi(\tau' \mid \tau'_{-i} = \tau_{-i})$$

and

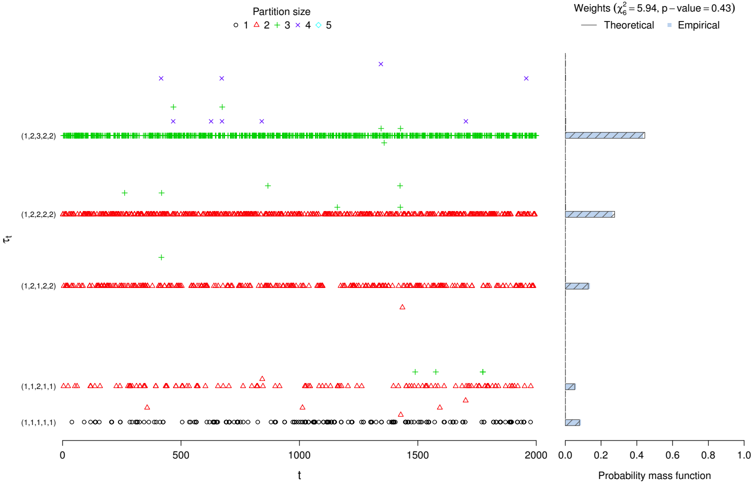
$$\pi(\tau' \mid \tau'_{-i} = \tau_{-i}) = \begin{cases} \frac{1}{C(\tau,i)} \prod_{j=1}^{|\tau'|} \omega_{\tau'_j} & \text{si } \tau'_{-i} = \tau_{-i} \\ 0 & \text{sinon} \end{cases}.$$

Gibbs sampler



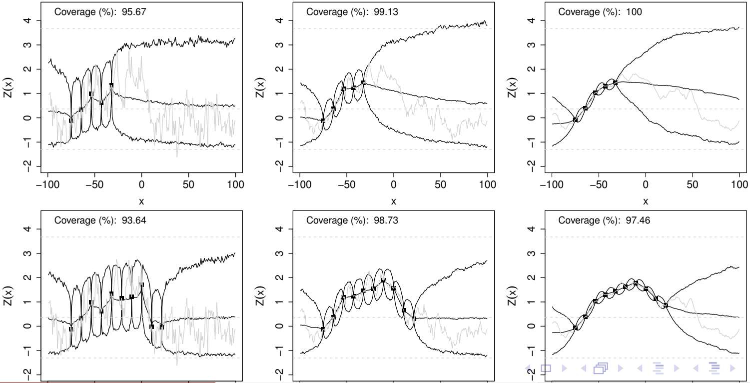
Gibbs sampler

Verification in the Brown-Resnick case with $k = 5$, $\sharp\mathcal{P}_5 = 52$



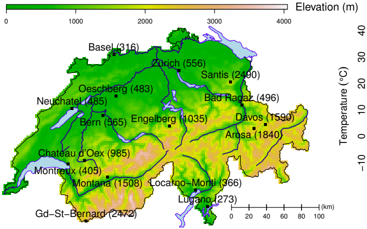
Conditional sampling

- Brown-Resnick process driven by FMB ($H = 1/4$, $1/2$ or $3/4$).
- Conditional sampling of a path (given $k = 5$ or 10 conditioning points).
- Conditional median and quantiles of order 0.025 and 0.975 evaluated numerically (confidence interval).



Application

- Annual maxima for temperatures in Switzerland at 16 stations :



- Particular interest for the heatwave from Summer 2003.

Application

- Max-stable model $\eta(x)$:

  ▷ Davison & Gholamrezaee ('11), Geostatistics of extremes. Proc. Roy. Soc. A.

  Marginal distributions

$$\eta(x) \sim \mathrm{GEV}(\gamma(x), \mu(x), \sigma(x)) \quad \text{avec} \begin{cases} \gamma(x) = \beta_{0,\gamma} \\ \mu(x) = \beta_{0,\mu} + \beta_{1,\mu} \mathrm{alt}(x) \\ \sigma(x) = \beta_{0,\sigma} + \beta_{1,\sigma} \mathrm{alt}(x) \end{cases} .$$

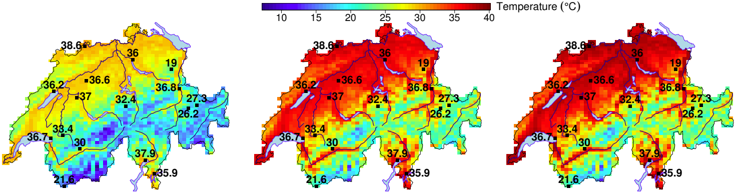
  Dependance structure of type "extremal Gaussian process" with correlation

$$\rho(x_1, x_2) = \exp\Big[- \Big(\frac{\|x_2 - x_1\|}{\lambda} \Big)^{\kappa} \Big].$$

- Model fitted by the "pairwise likelihood method" using annual maxima over the period 1965-2005.

Application

- Conditional sampling with values observed at the 16 stations in 2003.
- Estimation of conditional quantiles (0.025, 0.5, 0.975) :



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1 Introduction

2 Conditional distribution of max-stable random fields

3 Strong mixing properties of max-i.d. processes

Motivations

- Statistics of max-stable process based on non i.i.d. observations but rather on stationary weakly dependent observations.
- Recent results for ergodic and mixing properties of stationary max-stable and max-i.d. processes :

  ▷ Weintraub ('91) Sample and ergodic properties of some min-stable processes.

  ▷ Stoev ('10) Max-stable processes : representations, ergodic properties and statistical applications.

  ▷ Kabluchko & Schlather ('10) Ergodic properties of max-infinitely divisible processes.
- Ergodicity and mixing are important to derive strong law of large numbers but not enough to get central limit theorems.

Motivations

- CLTs for stationary weakly dependent processes are available under strong mixing assumptions.
- We consider here β-mixing (Volkonskii et Rozanov '59) : for random variables X_1, X_2,

$$\beta(X_1, X_2) = \|P_{(X_1,X_2)} - P_{X_1} \otimes P_{X_2}\|_{var}.$$

- Consider a continuous max-i.d. process η on a locally compact set T such that

$$\Big(\eta(t)\Big)_{t\in T} \overset{\mathcal{L}}{=} \Big(\bigvee_{\phi\in\Phi} \phi(t)\Big)_{t\in T}, \quad \text{with } \Phi \sim \mathrm{PPP}(\mu).$$

  with μ the exponent measure on $\mathcal{C}_0(T) = \mathcal{C}(T, [0, +\infty)) \setminus \{0\}$.
- For disjoint subsets $S_1, S_2 \subset T$, can we get an estimate for

$$\beta(S_1, S_2) = \beta(\eta_{|S_1}, \eta_{|S_2}) \quad ?$$

A natural decomposition of the point process

- For any $S \subset T$, $\Phi = \Phi_S^+ \cup \Phi_S^-$ with

$$\begin{aligned} \Phi_S^+ &= \{\phi \in \Phi;\ \exists s \in S,\ \phi(s) = \eta(s)\} \\ \Phi_S^- &= \{\phi \in \Phi;\ \forall s \in S,\ \phi(s) < \eta(s)\}. \end{aligned}$$

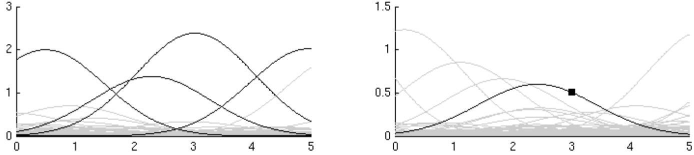

FIGURE: Realizations of the decomposition $\Phi = \Phi_S^+ \cup \Phi_S^-$, with $S = [0, 5]$ (left) or $S = \{3\}$ (right).

- Clearly for $s \in S$, $\eta(s) = \bigvee_{\phi \in \Phi_S^+} \phi(s)$ whence

$$\beta(S_1, S_2) \leq \beta(\Phi_{S_1}^+, \Phi_{S_2}^+).$$

A simple upper bound for $\beta(S_1, S_2)$

Theorem

The following upper bound holds true

$$\beta(S_1, S_2) \leq 2\,\mathbb{P}[\Phi^+_{S_1} \cap \Phi^+_{S_2} \neq \emptyset] \leq 2\int_{\mathcal{C}_0} \mathbb{P}[f \not<_{S_1} \eta,\, f \not<_{S_2} \eta]\,\mu(df).$$

In the particular case when η is a 1-Fréchet process,

$$\beta(S_1, S_2) \leq 2\,[C(S_1) + C(S_2)]\,[\theta(S_1) + \theta(S_2) - \theta(S_1 \cup S_2)]$$

with $C(S) = \mathbb{E}\big[\sup_S \eta^{-1}\big]$ and $\theta(S) = -\log \mathbb{P}\big[\sup_S \eta \leq 1\big]$ the areal coefficient (Coles & Tawn '96).

A CLT for max-stable random fields

Theorem

Let η be stationary simple max-stable on $\mathbb{Z}^d$ and define

$$X(h) = g(\eta(t_1 + h), \dots, \eta(t_p + h)), \quad h \in \mathbb{Z}^d.$$

Assume that there is $\delta > 0$ such that $\mathbb{E}[X(0)^{2+\delta}] < \infty$ and

$$2 - \theta(\eta(0), \eta(h)) = o(\|h\|^{-b}) \quad \text{for some } b > d \max\left(2, \frac{2+\delta}{\delta}\right).$$

Then $S_n = \sum_{\|h\| \leq n} X(h)$ satisfies the central limit theorem :

$$c_n^{-1/2}\left(S_n - \mathbb{E}[S_n]\right) \Longrightarrow \mathcal{N}(0, \sigma^2)$$

with $c_n = \mathrm{card}\{\|h\| \leq n\}$ and $\sigma^2 = \sum_{h \in \mathbb{Z}^d} \mathrm{cov}(X(0), X(h))$.

An application

- For a stationary 1-Fréchet random field on $\mathbb{Z}^d$, the extremal coefficient $\theta(h)$ gives an insight into the dependence structure.
- Recall that

$$\mathbb{P}[\eta(0) \leq y, \eta(h_0) \leq y] = \exp(-\theta(h_0)/y), \quad y > 0,$$

whence we deduce the naive estimator

$$\hat{\theta}_n^{(1)}(h_0) = -y \log \left(c_n^{-1} \textstyle\sum_{\|h\| \leq n} 1_{\{\eta(h) \leq y,\ \eta(h+h_0) \leq y\}} \right).$$

- To avoid the arbitrary choice of the arbitrary truncation level $y > 0$, Smith ('90) uses

$$\mathbb{E}\left[\min\left(\frac{1}{\eta(0)}, \frac{1}{\eta(h_0)} \right) \right] = \frac{1}{\theta(h_0)}$$

and suggests the estimator

$$\hat{\theta}_n^{(2)}(h_0) = \left(c_n^{-1} \textstyle\sum_{\|h\| \leq n} \min\left(\frac{1}{\eta(h)}, \frac{1}{\eta(h+h_0)} \right) \right)^{-1}.$$

An application

- Cooley, Naveau & Poncet ('06) suggest the use of the F-madogram

$$\mathbb{E}[|F(\eta(0)) - F(\eta(h_0))|] = \frac{1}{2}\frac{\theta(h_0)-1}{\theta(h_0)+1}$$

with $F(y) = exp(-1/y)$ and the alternative estimator

$$\hat{\theta}_n^{(3)}(h_0) = \frac{1 + 2c_n^{-1}\sum_{\|h\|\leq n}|F(\eta(h)) - F(\eta(h+h_0))|}{1 - 2c_n^{-1}\sum_{\|h\|\leq n}|F(\eta(h)) - F(\eta(h+h_0))|}.$$

Proposition

Assume that $\theta(h) = 2 + o(\|h\|^{-b})$ for some $b > 2d$.
Then, the estimators $\hat{\theta}_n^{(i)}(h_0)$ $(i = 1, 2, 3)$ are asymptotically normal :

$$c_n^{-1/2}\left(\hat{\theta}_n^{(i)}(h_0) - \theta(h_0)\right) \Longrightarrow \mathcal{N}(0, \sigma_i^2).$$

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